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iason-Nuant Optimal Cross-Pool Liquidity Provision Strategy

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Executive Summary

In this work, we tackle the problem of optimal Automated Market Making in Decentralized Crypto-Asset Liquidity Pools. We reviewed the approaches defined in the literature for defining an optimal strategy to be executed on a single liquidity pool, taking into account both rebalancing costs and impermanent losses. Hence, supposing the spot exchange rates to be driven by an elliptically distributed noise and exploiting their spectral properties, we derive an optimal multi-pool portfolio allocation method able to achieve two goals at the same time: diversify effectively the exposure of the agent's wealth and allow for the execution of the aforementioned strategy independently in each liquidity pool.

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Decentralized Finance (DeFi) has emerged as one of the most significant technological innovations of recent years. In this ecosystem, a wide range of crypto-assets has been introduced, together with cross-cryptocurrency exchange protocols that algorithmically regulate the interaction between supply and demand across different digital assets. Among these, Decentralized Exchange (DEX) protocols – such as Uniswap – rely on the presence of Liquidity Providers (LPs), who function as market makers. LPs supply liquidity by depositing pairs of cryptocurrencies into decentralized trading venues known as liquidity pools, adhering to a predefined set of protocol-specific rules.
Given the availability of multiple exchange rates and liquidity pools, an LP is faced with the problem of determining how to optimally allocate their capital across different pools. Thus, the liquidity provision decision can be framed as an optimal wealth allocation problem.
In this paper, we address this problem from two complementary perspectives. First, we analyze an optimal liquidity provision strategy tailored to cryptocurrency pairs traded under the Uniswap V3 protocol. Second, we consider liquidity provision strategies as investable assets and formulate a portfolio optimization problem at the LP level.
The structure of the paper is as follows. In Section 2, we review the optimal liquidity provision framework proposed in [3], which we adopt as a baseline model. In Section 3, we extend this framework to the setting in which an LP distributes capital across multiple liquidity pools. Section 4 introduces the concept of Agnostic Risk Parity portfolios, which provide more diversified allocations and reduce sensitivity to estimation errors in the covariance matrix of expected returns. Finally, in Section 5, we combine these elements to construct an optimal multi-pool liquidity provision strategy. A numerical implementation will be explored in future work.

1. Constant Function Markets and Concentrated Liquidity

1.1 Constant Function Markets (CFM)

Consider two cryptocurrencies X and Y. A liquidity pool is a decentralized trading venue in which LPs lock some amounts of asset X and asset Y to make them available for trading activity to any willing agent, called liquidity taker (from now on LT), at an exchange rate Z. The economic rationale for an LP to provide its assets is the possibility to earn fees as compensation for the liquidity provision activity.
A Constant Function Market (CFM) is a liquidity pool in which:

$$f(q_X, q_Y) = \kappa^2.$$

For some function f monotonically increasing both in q_X and in q_Y. This entails that the total amount q_X and q_Y of asset X and asset Y respectively locked into the pool is given by:

$$q_X = \varphi_\kappa(q_Y),$$

where φ_κ is a decreasing *level function* of the market function f at the *liquidity level* κ. Hence, whenever a LT wants to swap y units of asset Y, the amount x of asset X that he can obtain must verify the following relation:

$$f(q_X + (1-\tau)x, q_Y - y) = \kappa^2,$$

where τ represents the fee rate to be paid to LPs as compensation for their liquidity provision activity. As a result, the *execution exchange rate*, i.e. the exchange rate at which any liquidity taking trade is performed, is given by:

$$\tilde{Z}(y) = \frac{\varphi_\kappa(q_Y - y) - \varphi_\kappa(q_Y)}{(1-\tau)y}.$$

Similarly, whenever a LT wants to swap x units of asset X the amount y of asset Y that he can obtain satisfies:

$$f(q_X - x, q_Y + (1-\tau)y) = \kappa^2,$$

and the consequent execution exchange rate is given by:

$$\tilde{Z}(-y) = \frac{\varphi_\kappa(q_Y) - \varphi_\kappa(q_Y + (1-\tau)y)}{y}.$$

Hence, the spot exchange rate quoted on the liquidity pool is given by:

$$Z = \lim_{y \to 0} Z(y) = -\frac{1}{1-\tau}\varphi'_\kappa(q_Y).$$

On the other hand, liquidity provision should not impact the spot exchange rate. Hence, when liquidity provision is performed by LPs:

$$Z = \frac{q_X}{q_Y} = \frac{q_X + x}{q_Y + y}.$$

In the case of the UniswapV2 protocol, one of the most used Constant Function Market:

$$f(q_X, q_Y) := q_X q_Y = \kappa^2, \qquad \varphi_\kappa(q_Y) = \frac{\kappa^2}{q_Y}, \qquad Z = \frac{q_X}{q_Y}.$$

1.2 CFMs with Concentrated Liquidity (CFM-CL)

In CFMs, LPs are forced to offer liquidity at any possible exchange rate, determined by the dynamics of LT transactions. Constant Function Markets with Concentrated Liquidity, on the other hand, adds the possibility for LPs to select specific price boundaries to provide their liquidity within, whose bounds should be chosen among a set of discrete *price ticks* $\{Z^1, Z^2, ..., Z^n\}$. The smallest range $(Z^i, Z^{i+1}]$ is called *tick range*.
Let the liquidity provider choose a price interval $(Z_l, Z_u]$. For any chosen range, the LPâs asset amounts x and y are specified by key formulae, which determine what assets the LP holds depending on where the pool price currently sits relative to the chosen boundaries:

$$\begin{cases} x = 0 & \text{if } Z \le Z^\ell \\ x = \tilde{\kappa}\left(Z^{1/2} - (Z^\ell)^{1/2}\right) & \text{if } Z^\ell < Z \le Z^u \\ x = \tilde{\kappa}\left((Z^u)^{1/2} - (Z^\ell)^{1/2}\right) & \text{if } Z > Z^u \end{cases},$$

$$\begin{cases} y = \tilde{\kappa}\left((Z^\ell)^{-1/2} - (Z^u)^{-1/2}\right) & \text{if } Z \le Z^\ell \\ y = \tilde{\kappa}\left(Z^{-1/2} - (Z^u)^{-1/2}\right) & \text{if } Z^\ell < Z \le Z^u \\ y = 0 & \text{if } Z > Z^u \end{cases}.$$

This implies that:

- If $Z \le Z_l$ the LP only holds Y;
- If $Z > Z_u$ the LP only holds X.

These formulae dictate how the LP's holdings evolve within the range $(Z^l, Z^u]$[1] and what they receive upon exiting the pool.
Liquidity depth κ within a tick range $(Z^l, Z^u]$ is constant unless more liquidity is deposited or withdrawn. When the marginal price crosses any of the tick boundaries, the pool must execute distinct trades using the appropriate depth for each tick. Aggregate liquidity depth in a given tick is determined by summing the depths of all LP positions in that range.

$$\kappa = \sum_i \tilde{\kappa}_i.$$

Let p the total amount of fees of the pool. The amount of fees earned by an LP is proportional to the ratio between the LP's share of liquidity $\tilde{\kappa}$ and the total liquidity of the pool κ:

$$\tilde{p} = \frac{\tilde{\kappa}}{\kappa} p \cdot \mathbb{1}_{Z_\ell < Z \leq Z_u}.$$

This implies that LPs who concentrate more liquidity in narrower ticks capture a greater share of fees in the range selected $(Z^l, Z^u]$, but this increases the concentration risk, i.e. the chance that the price leaves the tick range letting the LP liquidity switch from being *active* (i.e. available for LT transactions) to being *inactive* (i.e. unavailable for LT transactions).
UniswapV3 is an example of CFM with CL and will be the protocol in which the analysis of the liquidity provision strategies introduced in the next sections will be carried out.

2. Optimal Liquidity Provision: Case of Single Pool

2.1 Basic Assumptions

Two assets X and Y are assumed to be exchanged in a CFM with CL. Each LP must choose a range $(Z^l, Z^u]$ such that he is willing to buy or sell to any LT an amount x of the first or y second asset at an exchange rate within the bounds Z^l, Z^u of the range. Such bounds must be chosen by the LP in a grid $\{Z_i\}_{i=1}^N$.
The marginal exchange rate Z_t expressing the amount of asset X received per each unit of asset Y is assumed to follow the stochastic dynamics given by:

$$dZ_t = \mu_t Z_t dt + \sigma Z_t dW_t,$$

where:

- μ_t: Drift process with finite fourth moment;
- σ: Volatility coefficient, assumed to be constant;
- W_t: Standard Brownian motion.

The quantities x and y of assets X and Y allocated by an LP can be expressed in terms of the depth of the LP liquidity in the pool κ, i.e. the percentage of assets X and Y allocated by the LP over the whole amount allocated by all the LPs in the pool measured in a reference numeraire (for instance USDC):

$$x_t = \kappa \left((Z_t^u)^{1/2} - (Z_t^\ell)^{1/2} \right), \quad y_t = \kappa \left((Z_t^\ell)^{-1/2} - (Z_t^u)^{-1/2} \right).$$

[1] A LP can quote several ranges at the same time.

2.2 Wealth Dynamics

In [3], the authors develop an optimal strategy for dynamically managing liquidity in a CFM. The goal is to maximize the LP's terminal wealth, which includes fees earned from trades and P&L from market-making. The strategy adjusts the liquidity range and skew based on the DEX spot.
For log-utility preferences, they derive an explicit solution that navigates the trade-off between collecting fees and managing impermanent loss. When volatility rises, the LP broaden the liquidity range to prevent the possibility of seeing her liquidity inactive. In extreme cases of high volatility, the LP may exit the pool entirely as liquidity provisioning can become unprofitable. On the other hand, increased fee potential due to higher trading activity encourages concentrating liquidity in a narrow band around the exchange rate, balanced against the risks of range concentration. When the marginal exchange rate exhibits stochastic drift, the strategy shifts liquidity placement to capture trading flows and capitalize on expected rate changes.
The LP's wealth, expressed in units of asset X, can be decomposed additively:

$$\tilde{x}_t = \alpha_t + p_t + c_t,$$

where the processes α_t, p_t and c_t represent:

1. **Position Value** (α_t): total asset value of the LP in the liquidity pool;
2. **Fee Revenue** (p_t): income from fees earned by the LP and paid by LTs;
3. **Rebalancing Costs** (c_t): Costs incurred while adjusting liquidity ranges.

In the following we provide a brief description of each aforementioned component.

2.3 Position Value

The position value α_t evolves according to the following stochastic differential equation:

$$d\alpha_t = \frac{\tilde{x}_t}{\delta_\ell + \delta_u}\left(-\frac{\sigma^2}{2}dt + \mu_t\delta_u dt + \sigma\delta_u dW_t\right) = d\text{PL}_t + \frac{\tilde{x}_t}{\delta_\ell + \delta_u}\left(\mu_t\delta_u dt + \sigma\delta_u dW_t\right),$$

where:

- $\tilde{x}_t$: the LP's wealth in the reference numeraire;
- δ_u and δ_ℓ: control parameters defining the upper and lower bounds of the liquidity range, such that the liquidity range is defined like:

$$\begin{cases} (Z_t^u)^{1/2} & = Z_t^{1/2}/(1-\delta_t^u/2), \\ (Z_t^l)^{1/2} & = Z_t^{1/2}(1-\delta_t^l/2). \end{cases} \tag{1}$$

The deterministic component:

$$d\text{PL}_t = -\frac{\sigma^2}{2}\cdot\frac{\tilde{x}_t}{\delta_\ell + \delta_u}dt, \tag{2}$$

represents the **predictable loss**, i.e. losses due to arbitrages made available between the liquidity range quoted in the pool by the LP and other trading venues due to the latency

between the time of variation in the exchange rate on other markets and the time of update of liquidity provision by the LP in the liquidity pool, known as **loss versus rebalancing** (LVR).
On the other hand, due to the observability of the drift component, the LP can skew the liquidity provision range upward or downward according to whether μ_t is positive or negative respectively in order to capture a larger share of fee revenues (see the next Section 2.4).
Hence, the LP faces a trade-off between a **drift-based strategic positioning** and a drag component scaling in time like the variance of the exchange rate caused by arbitrages.

2.4 Fee Revenue

The dynamics of p_t expresses how the fee generation process, adjusted for spread and concentration risk, accrues over time, hence is given by:

$$dp_t = \frac{4}{\delta_\ell + \delta_u} \pi_t \tilde{x}_t \, dt - \gamma \left(\frac{1}{\delta_\ell + \delta_u} \right)^2 \tilde{x}_t \, dt, \tag{3}$$

where:

- $\delta_\ell + \delta_u = \delta_t$: the total spread of the liquidity position.
- π_t: the pool's fee rate, i.e. the rate of fees generated instantaneously by transactions executed in the pool.
- $\gamma > 0$: the concentration cost parameter, i.e an instantaneous (constant) friction coefficient penalizing the overall profitability of the LP activity according to the squared inverse of the liquidity range length.

As shown in (3), the fee revenue dynamics is made up of two distinct additive components:

- **Fee Revenue**:
$$\frac{4}{\delta_\ell + \delta_u} \pi_t \tilde{x}_t \, dt.$$
This term captures the proportional relationship between the LP's position size and the pool's profitability rate π_t. Narrower spreads (δ_t) yield higher fee income per trade.

- **Concentration Risk Penalty**:
$$-\gamma \left(\frac{1}{\delta_\ell + \delta_u} \right)^2 \tilde{x}_t \, dt.$$
This term accounts for the risk of the marginal rate Z_t exiting the LP's liquidity range, reducing the effectiveness of fee collection for smaller spreads.

Hence, similarly to the position value, the fee revenue captures a trade-off, namely the trade-off between profitability coming from the liquidity provision activity and concentration cost: the narrower the length of the liquidity range $\delta_t = \delta_t^l + \delta_t^u$, the higher the fee rate captured by the LP in the range $(\delta_t^l, \delta_t^u]$ but the higher cost-opportunity coming from quoting in a narrower range.

2.5 Rebalancing Cost

Finally, c_t, represents the rebalancing costs incurred by the LP as they adjust their liquidity position. Such costs are proportional to the holdings of asset Y that need adjustment and are given by:

$$dc_t = -\zeta \cdot \frac{\delta_u}{\delta_\ell + \delta_u} \tilde{x}_t \, dt, \tag{4}$$

where:

- ζ: a constant representing the proportional execution cost associated with rebalancing the liquidity position.

2.6 Optimal Liquidity Strategy

At this point, we introduce the position asymmetry function:

$$\rho_t = \frac{\delta_t^u}{\delta_t^l + \delta_t^u}, \tag{5}$$

which measures how much the position is skewed, i.e. how much the current spot Z_t is not equidistant from the boundaries $(Z_t^l, Z_t^u]$. In particular we have that:

- If $\rho = \frac{1}{2}$ the range is perfectly centered;
- If $\rho \to 0$ then $\delta_t^u \to 0$, hence $Z_t^u \to Z_t$: only *token1* is offered in the position $(Z_t^l, Z_t^u]$;
- If $\rho \to 1$ then $\delta_t^l \to 0$, hence $Z_t^l \to Z_t$: only *token0* is offered in the position $(Z_t^l, Z_t^u]$.

The author of [3] assume that:

$$\rho_t = \rho(\delta_t, \mu_t) = \frac{1}{2} + \frac{\mu_t}{\delta_t} = \frac{1}{2} + \frac{\mu_t}{\delta_t^l + \delta_t^u} \quad \forall t \in [0, T]. \tag{6}$$

In order to gain fees from the liquidity provision activity, the LP must keep the provided liquidity active, i.e. within a range enclosing the current spot price Z_t. Hence:

- If $\mu_t > 0$ the spot tends to move towards Z_t^u and the LP adjusts the liquidity range to the right;
- If $\mu_t < 0$ the spot tends to move towards Z_t^l and the LP adjusts the liquidity range to the left.

The LP wants to optimize the final expected log-utility of its capital locked into the pool. It can be proved that, assuming expression (6) for the asymmetry function, the final expected log-utility problem can be formulated with just one control variable, given by the liquidity provision range length δ_t. Moreover, we need to assume that the asymmetry function is square-integrable and this entails that δ_s must be such that:

$$\mathcal{A}_t = \left\{ \delta_s : [t, T] \to \mathbb{R} : \quad \delta_s \ is \ \mathbb{F}_t - adapted, \quad \int_t^T \delta_s^{-4} ds < \infty \quad \mathbb{P} - a.s. \right\}.$$

Hence, under the square-integrability assumption of the asymmetry function and assuming the following **profitability condition**:

$$\pi_t \geq \eta_t = \frac{\sigma^2}{8} - \frac{\mu_t}{4}\left(\mu_t - \frac{\sigma^2}{2}\right),$$

the stochastic dynamic control problem can be formulated as:

$$\sup_{\delta \in \mathcal{A}_t} u^{\delta}(t, \tilde{x}, z, \pi, \mu) = \mathbb{E}_{t,\tilde{x},z,\pi,\mu}\left[\log(x_T^{\delta})\right]. \tag{7}$$

Under the previous assumptions, the resulting Hamilton-Jacobi-Bellman equations have a unique solution **optimal spread** given by:

$$\delta^* = \frac{2\gamma + \mu^2\sigma^2}{4\pi - \frac{\sigma^2}{2} + \mu\left(\mu - \frac{\sigma^2}{2}\right)}. \tag{8}$$

Once the trading frequency has been decided, the strategy consists in rebalancing the liquidity around the bounds defined by (1), with δ_u and δ_l defined by:

$$\begin{aligned} \delta_l &= \delta^*/2 - \mu_t, \\ \delta_u &= \delta^*/2 + \mu_t. \end{aligned} \tag{9}$$

It can be seen that the optimal range is strictly increasing in the volatility of the spot rate, so that the agent widens her liquidity bounds in highly volatile scenarios, to avoid the possibility of being left with inactive liquidity between the rebalancing times.
A higher fee rate, on the other hand, encourages the agent to narrow her position, in order to capture a larger portion of the trading fees.

2.7 Simulation Results

We assessed the performance of the aforementioned strategy over a set of 1000 simulations of the spot exchange rate dynamics. We simulated a WETH-USDT pool assuming:

- A realistic level of volume/day ($\sim$ 20 MIO);
- Gas fees at 25 USDT/transaction;
- 1 block added every $\sim$ 14 seconds;
- Arbitrageurs taking arbitrage opportunities across the simulated WETH/USDT DEX and the corresponding CEX.

The CEX spot exchange drift and the volatility coefficients are set to $\mu = 50\%$ and $\sigma = 50\%$ (1:1 Sharpe Ratio of the Buy-and-Hold strategy assuming no risk-free rate).
We tested the optimal AMM strategy outlined versus a simple Spot-Tracking strategy, keeping a liquidity provision range centered a +/-5% of the current spot price and adjusting it whenever the DEX spot price go out of such the corresponding boundaries. Both strategies were implemented with daily monitoring (i.e. at most 1 adjustment/day). Finally, we assumed the fee rate to be a martingale. Analyzing fig.(1), we notice that the Optimal Strategy is much more reactive than the Spot-Tracking one, being dependent also on the estimated fee rate. In facts, the absolute impermanent loss, defined as the difference between the capital value locked in the pool at time t and the capital value locked in the pool at the time of the last rebalancing $t - \Delta t$, where Δt is the rebalancing frequency (1 day in this case), is expected to be close to 0 at the rebalancing time whereas the spot-tracking case shows a clear negative drift, sign of a much lower probability of adjusting the liquidity at the rebalancing time.
Moreover, analyzing the expected liquidity and active liquidity (i.e. the portion of liquidity provided in a range including the spot, hence available for swap at time t, fig.(2)), it is clear

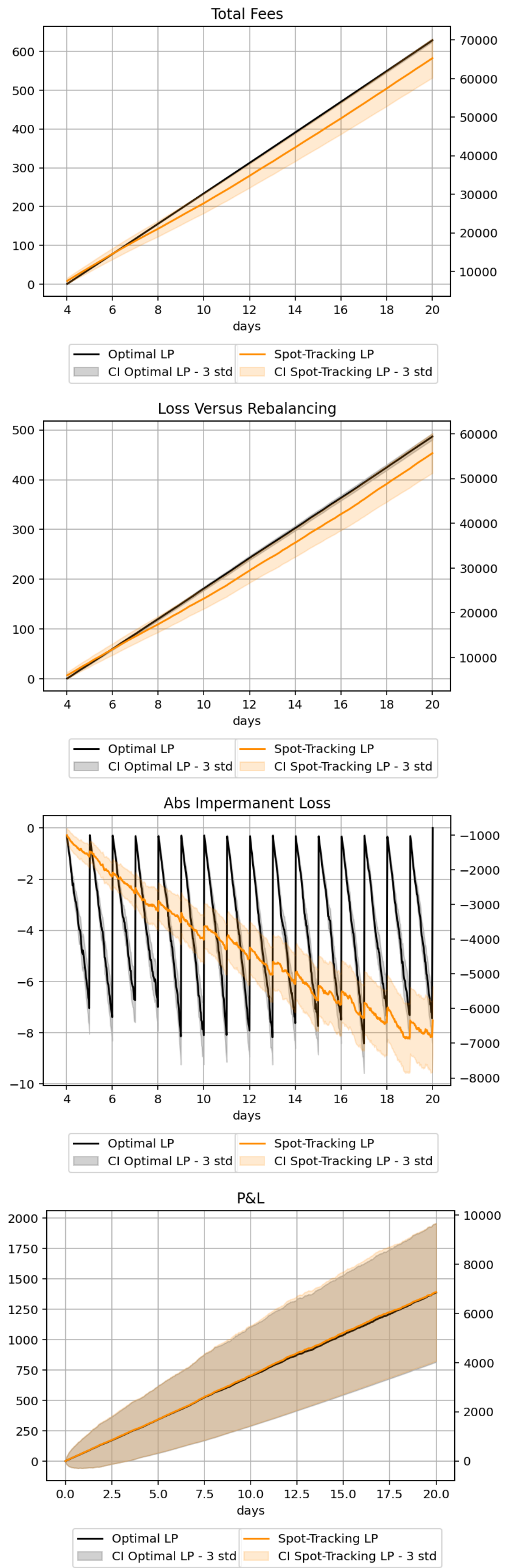


FIGURE 1: *Total Fees (first), losses-vs-rebalancing (LVR, second), absolute impermanent losses (third) and total P&L (fourth).*

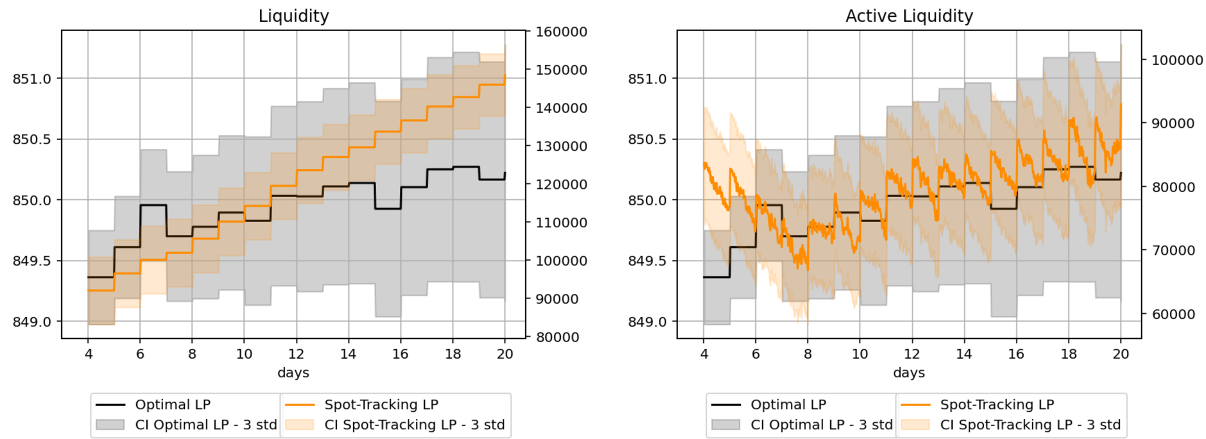


FIGURE 2: Liquidity (left) vs active liquidity (right).

why total fees and losses vs rebalancing are 2 order of magnitude lower for the Optimal LP vs the Spot-Tracking LP: liquidity and active liquidity show the same ratio approximately. Finally, what said about the absolute impermanent losses is confirmed: the Optimal Strategy keeps all its expected liquidity active through the whole period, whereas the Spot-Tracking strategy has active just a portion of the provided liquidity.
Analyzing the cumulated expected PnL (in base 100), the two strategies look equivalent: both the expected PnL and the confidence interval bounds almost overlap. Also the Sharpe Ratios look similar: noisy in both cases and averaging 0 in time.

Finally, we stress the fact that the overall profitability is sensitive to the assumptions made about the data generating process of the daily transactions, which determinate volumes, the level of gas fees and the initial amount of capital allocated to the pool: in other simulations with lower volumes and lower initial capital profitability decreases (not reported here).

2.8 Backtest and Live Results

We performed backtests on historical spot exchange rate. In this case we tested the Optimal LP versus both the Spot-tracking LP and the Static LP (i.e. a LP that simply put liquidity at the start period and doesnât adjust the liquidity range).
We estimated the drift parameter by a 20-day simple moving average and the volatility parameter by 20-day rolling standard deviation of the log-returns of the spot. The evidence (see fig. (4)) follows:

- LVR overcome Total fees;
- Active liquidity is heavily clustered: after the cluster at the start of the backtest period, the optimal strategy according to (8) is not to take part into liquidity provisioning activities since the profitability condition is not met;
- The total PnL is negative and follows a path similar to the spot: this is explained by the fact that for most of the time the liquidity is inactive and so the PnL is mostly driven by the capital loss due to the spot decline and not by fees, which is actually the economic reason to provide liquidity in a DEX.

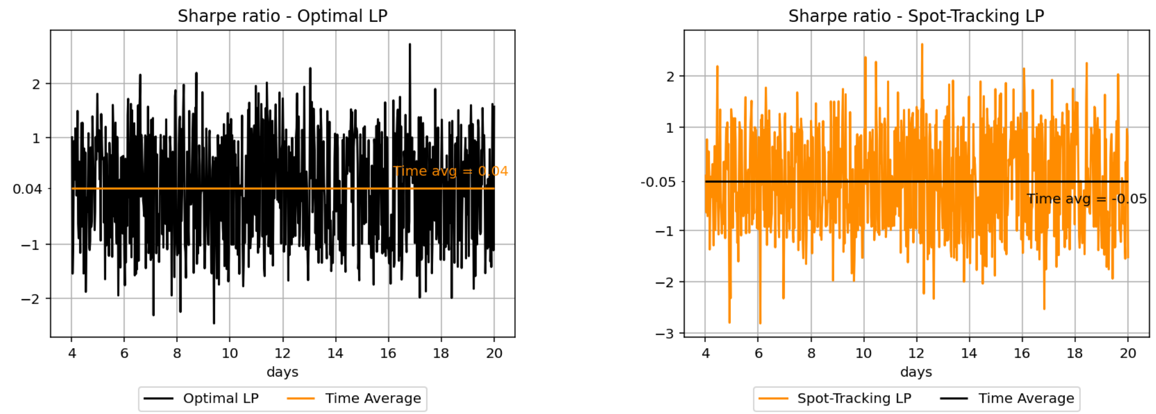


FIGURE 3: *Simulated Sharpe Ratio of the Optimal LP (left) vs simulated Sharpe Ratio of the Spot-Tracking LP (right).*

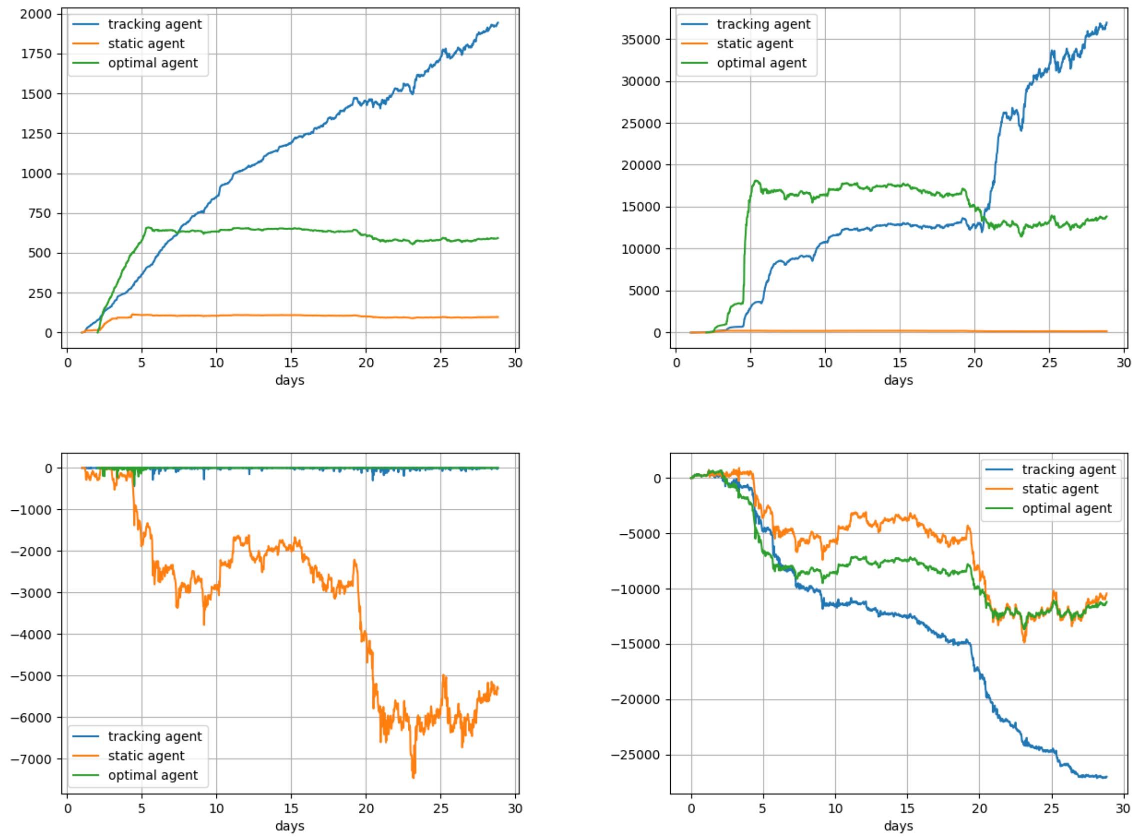
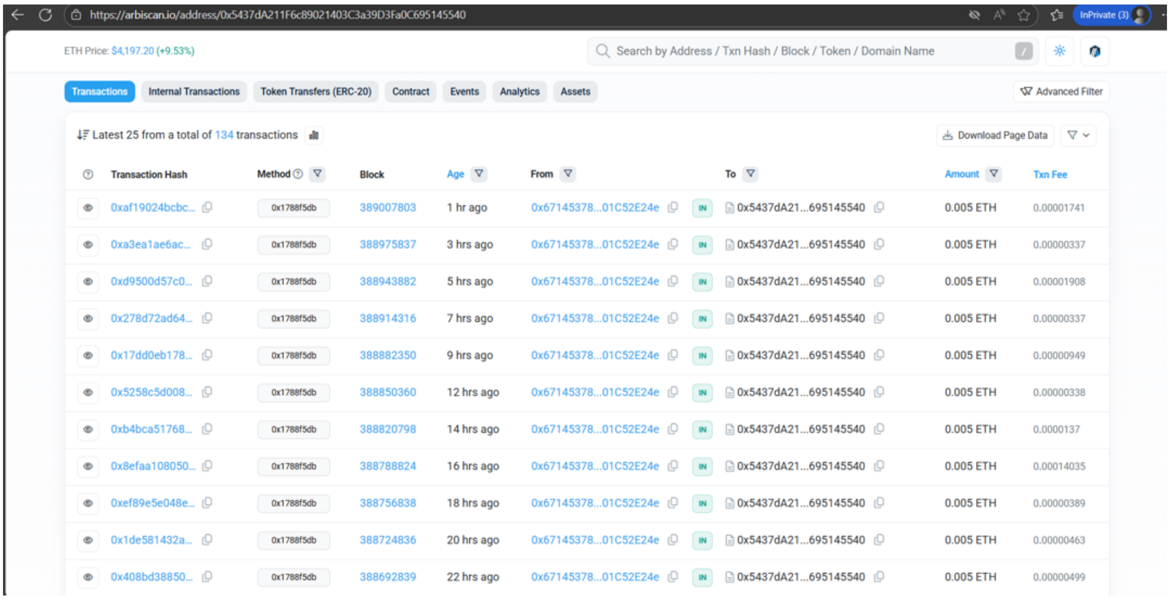


FIGURE 4: *Total Fees (upper-left), losses-vs-rebalancing (LVR, upper-right), absolute impermanent losses (lower-left) and total P&L (lower-right) of the backtested strategies.*

Similar results were obtained in other time windows.
Finally, as far as the live test is concerned, our partner Nuant A.G. deployed a smart contract tracking faithfully the performance of the mono-pool strategy starting from October 1st on the UniswapV4 blockchain. The strategy is not profitable so far (see fig.5).


https://arbiscan.io/address/0x5437dA211F6c89021403C3a39D3Fa0C695145540

ETH Price: $4,197.20 (+9.53%)

Transactions | Internal Transactions | Token Transfers (ERC-20) | Contract | Events | Analytics | Assets

Latest 25 from a total of 134 transactions

Transaction Hash	Method	Block	Age	From		To	Amount	Txn Fee
0xaf19024bcbc...	0x1788f5db	389007803	1 hr ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00001741
0xa3ea1ae6ac...	0x1788f5db	388975837	3 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00000337
0xd9500d57c0...	0x1788f5db	388943882	5 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00001908
0x278d72ad64...	0x1788f5db	388914316	7 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00000337
0x17dd0eb178...	0x1788f5db	388882350	9 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00000949
0x5258c5d008...	0x1788f5db	388850360	12 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00000338
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0x408bd38850...	0x1788f5db	388692839	22 hrs ago	0x67145378...01C52E24e	IN	0x5437dA21...695145540	0.005 ETH	0.00000499

FIGURE 5: Transactions performed by the Smart Contract deployed by Nuant A.G. on UniswapV4 tracking the optimal strategy.

3. Optimal Liquidity Provision: Case of Multiple Pools

3.1 Multiple Spot Exchange Rate Dynamics

Let's extend the analysis to the case of multiple exchange rates available.
Let W_t a d-dimensional standard Wiener process and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ the filtered space such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of W_t. Let Z_t be a d-dimensional Ito process representing the exchange rate of a set of n tokens, hereafter $token_i$, $\forall i \in \{1, 2, ..., n\}$ with respect to another reference token, hereafter referenced as $token_0$ and let:

$$dZ_t = \mu I Z_t dt + \sigma I Z_t dW_t, \tag{10}$$

where $\mu \in \mathbb{R}^n$ is the vector of real coefficients, $\sigma \in \mathbb{M}_{n\times n}(\mathbb{R})$ is a matrix such that $\Sigma := \sigma\sigma^T$ is a positive definite matrix and $I \in \mathbb{M}_{n\times n}$ the $n \times n$ identity matrix, be the stochastic differential equation describing the dynamics of X_t. This entails the following dynamics of log-returns $\{X_t\}_{t\geq 0}$:

$$dX_t = \left(\mu - \frac{1}{2}\psi\right)dt + \sigma dW_t, \tag{11}$$

where:

$$\psi := diag(\Sigma), \tag{12}$$

given $diag : \mathbb{M}_{\mathbb{R}(n\times n)} \to \mathbb{R}^n$ the operator which yields the main diagonal of a matrix.

By integrating eq. (11) in an interval $[0, t]$ we have that:

$$X_t = X_0 + \left(\mu - \frac{1}{2}\psi\right)t + \sigma W_t. \tag{13}$$

Hence:

$$Var(X_t) = \sigma Var(W_t)\sigma^T = \sigma(\mathcal{I}t)\sigma^T = \sigma\sigma^T t = \Sigma t, \tag{14}$$

where $\mathcal{I} \in \mathbb{M}_{n\times n}(\mathbb{R})$ is the $n \times n$ identity matrix.

3.2 Reformulation in the Principal Component Space...

Since Σ is a positive definite matrix by construction there exist $\Lambda, V \in \mathbb{M}_{n\times n}(\mathbb{R})$ matrix of eigenvalues and eigenvectors of Σ such that:

$$\Sigma = V\Lambda V^T = V\sqrt{\Lambda}\sqrt{\Lambda}V^T. \tag{15}$$

Recalling that, by definition, $\Sigma = \sigma\sigma^T$, it follows that:

$$\sigma = V\sqrt{\Lambda}. \tag{16}$$

Moreover, since $\sqrt{\Lambda} = \left(\sqrt{\Lambda}\right)^T$:

$$\sigma = V\sqrt{\Lambda} = \left(\sqrt{\Lambda}\right)^T V^T = \sqrt{\Lambda}V^T. \tag{17}$$

Hence, we can apply a suitable rotation to the original process of log-returns X_t and obtain a process $\tilde{X}_t$ such that:

$$\tilde{X}_t = V^T X_t = V^T\left(\mu - \frac{1}{2}\psi\right)dt + V^T V\sqrt{\Lambda}dW_t = \left(\tilde{\mu} - \frac{1}{2}\tilde{\psi}\right)dt + \sqrt{\Lambda}dW_t, \tag{18}$$

where:

$$\tilde{\mu} := V^T\mu, \tag{19}$$

and:

$$\tilde{\psi} := V^T diag(\Sigma) = V^T diag(V\Lambda V^T). \tag{20}$$

This paves the way leading to the possibility to recover the 1-dimensional optimal liquidity-providing strategy independently for each exchange rate by simply rotating the original log-return process X_t to obtain the process $\hat{X}_t$ representing the log-returns in an underlying-risk-factor space.
This is due to the fact that the optimal strategy developed in [3] is determined by maximizing an expected value and, hence, it is agnostic of the precise path followed by the driving process (either the original Wiener process W_t or its rotation $\hat{W}_t = V^T W_t$). Let the value process of the agent's portfolio at time t:

$$\mathcal{V}(t, X_t) := \sum_{i=1}^{n} z_i(t)\tilde{x}_i(t), \tag{21}$$

where $\tilde{x}_i(t)$ is the value process of the agent's wealth locked into a fictious liquidity pool exchanging the i-th underlying risk factor vs $token_0$:

$$d\tilde{x}_i(t) = \tilde{x}_i(t)\left(\frac{1}{\tilde{\delta}^l_{t,i} + \tilde{\delta}^u_{t,i}}\right)\left[\left(4\tilde{\pi}_{t,i} - \frac{\tilde{\psi}_i}{2} + \tilde{\mu}_i\tilde{\delta}^u_{t,i}\right)dt + \sqrt{\lambda_i}\tilde{\delta}^u_{t,i}dW_t\right] - \gamma_i\left(\frac{1}{\tilde{\delta}^l_{t,i} + \tilde{\delta}^u_{t,i}}\right)^2\tilde{x}_i(t)dt, \tag{22}$$

where $\tilde{\pi}_{t,i}$ and $\tilde{\gamma}_i$ represent the fee rate and the concentration cost parameter for the exchange rate of the i-th underlying risk factor vs $token_0$, analogous to those defined in [3]

for the exchange rate of $token_i$ vs $token_0$, $\forall i \in \{1,2,...,n\}$: the formers will be estimated conditioned to the latters as it will be explained in the next paragraph.
Now, by Ito's lemma, it follows that the log-wealth-process locked in the LP for the i-th underlying risk factor is given by:

$$\begin{aligned} d\log\tilde{x}_{t,i} = \frac{1}{\tilde{x}_{t,i}}d\tilde{x}_{t,i} - \frac{1}{2\tilde{x}^2_{t,i}}d<\tilde{x}_{t,i}>^2_t = \\ \left(\frac{1}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right)\left[4\tilde{\pi}_{t,i} - \frac{\tilde{\psi}_i}{2} + \left(\frac{\tilde{\delta}^u_{t,i}}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right)\frac{\tilde{\psi}_i}{2} - \left(\tilde{\gamma}_i + \frac{\lambda_i}{2}(\tilde{\delta}^u_{t,i})^2\right)\left(\frac{1}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right)\right]dt + \\ \left(\frac{\tilde{\delta}^u_{t,i}}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right)\left[\left(\tilde{\mu}_i - \frac{1}{2}\tilde{\psi}_i\right)dt + \sqrt{\lambda_i}dW_t\right] = \\ \tilde{c}_i(\tilde{\pi}_{t,i}) + \left(\frac{\tilde{\delta}^u_{t,i}}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right)d\tilde{X}_t \quad \forall i \in \{1,2,...n\}. \end{aligned} \tag{23}$$

Note that $\tilde{\delta}^u_{t,i}$ and $\tilde{\delta}^l_{t,i}$ are a function of the fee rate $\tilde{\pi}_{t,i}$ (see eq. 27 in [3]) and term $\tilde{c}_i(\pi_{t,i})$ in turn depends on them, other than $\tilde{\mu}_i$, $\tilde{\psi}_i$ λ_i and $\tilde{\gamma}_i$: the latter are assumed to be constant, hence the notation $\tilde{c}_i(\tilde{\pi}_{t,i})$.
In vector notation, let $d\tilde{\mathcal{X}}_t := [d\log\tilde{x}_{t,i}]^n_{i=1}$:

$$d\tilde{\mathcal{X}}_t = \tilde{c}(\tilde{\pi}_t) + \tilde{\Delta}^u_t d\tilde{X}_t, \tag{24}$$

with $\tilde{\Delta}^u_t \in \mathbb{M}_{\mathbb{R}}(n\times n)$ diagonal matrix containing the sequence $\left\{\frac{\tilde{\delta}^u_{t,i}}{\tilde{\delta}^l_{t,i}+\tilde{\delta}^u_{t,i}}\right\}^n_{i=1}$ on the main diagonal, $\tilde{\pi}_t \in \mathbb{R}^n$ vector of the fee rates $\{\tilde{\pi}_{t,i}\}^n_{i=1}$ and $\tilde{c}(\tilde{\pi}_t) \in \mathbb{R}^n$ containing the sequence $\{\tilde{c}_i(\pi_{t,i})\}^n_{i=1}$.
Note that the fee rates $\tilde{\pi}_t$ and the concentration cost parameters $\tilde{\gamma}$ of the pools in the fundamental risk factor space are unobservable. However, as it will be made clear, we will assume that they depend exclusively on the fee rate π_t in the real asset space. Moreover, the strategy is specified in terms of the liquidity provision policy $(\tilde{\delta}^u_t, \tilde{\delta}^u_t)$, which should dictate the policy in the asset space.
Hence, define the conditional expectation with respect to the σ-algebra generated by the fee rate as estimator of the future expected returns $\tilde{\pi}_t$, that is:

$$\tilde{p} := \mathbb{E}[d\tilde{\mathcal{X}}_t|\tilde{\pi}_t]. \tag{25}$$

Now, let:

$$\mu_{\tilde{p}} := \mathbb{E}[\tilde{p}] = \left[\mathbb{E}[\tilde{c}(\tilde{\pi}_t)] + \tilde{\Delta}^u_t\left(\tilde{\mu} - \frac{1}{2}\tilde{\psi}\right)\right]dt, \tag{26}$$

and:

$$\Sigma_{\tilde{p}} := Var(\tilde{p}) = \left(\tilde{\Delta}^u_t\right)^2\Lambda dt. \tag{27}$$

Since $\tilde{\delta}^l_t$ and $\tilde{\delta}^u_t$ are also a function of the vector of the fee rates $\tilde{\pi}_t$ (see eq. (9)), we have that[2]:

$$\tilde{p} \sim \mathcal{N}(\mu_{\tilde{p}}, \Sigma_{\tilde{p}}). \tag{28}$$

[2]We assumed a Gaussian driving process. The same would happen in case of any elliptically distributed driving process.

3.3 ...and Back into the Asset Space

The linearity of both the conditional expectation and transformation between the underlying risk factor space and the asset space causes the conditional expectation of the log-wealth process in the asset space to be straightforward:

$$p := \mathbb{E}[d\mathcal{X}_t|\tilde{\pi}_t] = \mathbb{E}[V^T d\tilde{\mathcal{X}}_t|\tilde{\pi}_t]. \tag{29}$$

Hence, it's mean and covariance matrix are respectively:

$$\mu_p := \mathbb{E}[p] = \left[\mathbb{E}[V^T \tilde{c}(\tilde{\pi}_t)] + \tilde{\Delta}^u_t\left(\mu - \frac{1}{2}\psi\right)\right]dt, \tag{30}$$

and:

$$\Sigma_p := Var(p) = (\tilde{\Delta}^u_t)^2 \Sigma dt. \tag{31}$$

We stress that the latter passage is of paramount importance, since the allocation can be made just selecting weights expressed as percentage of the total wealth available to the LP in the numÃ©raire asset (i.e. in the asset space, other than in the underlying risk factor space): this will be clear in the next section (sec. [4]).
At this point, we need to estimate the unobservable parameters $\tilde{\pi}_t$ and $\tilde{\gamma}$ and the optimal policy in the asset space. As said before, we assume that the fee rate in the fictious pool of the i-th risk factor $\tilde{\pi}_{i,t}$ depends just on the fee rate vector π, hence the sigma-algebra generated by $\tilde{\pi}$ is equal to the σ-algebra generated by π. Moreover:

$$p = V^T \tilde{p}. \tag{32}$$

Hence, it must hold that:

$$\begin{cases} \mathbb{E}[p|\tilde{\pi}_t] = \mathbb{E}[V^T \tilde{p}|\tilde{\pi}_t] = \mathbb{E}[V^T \tilde{p}|\pi_t] = \mathbb{E}[p|\pi_t], \\ Var[p|\tilde{\pi}_t] = Var[V^T \tilde{p}|\tilde{\pi}_t] = Var[V^T \tilde{p}|\pi_t] = Var[p|\pi_t]. \end{cases} \tag{33}$$

In our implementation of the optimal strategy described in [3], the fee rate $\tilde{\pi}_{t,i}$ is estimated pathwise. Hence, a reasonable way to estimate the unobservable parameters and establish the dependence of the optimal allocation policy in the asset space (δ^l_t, δ^u_t) from the optimal allocation policy in the fundamental risk factor space $(\tilde{\delta}^l_t, \tilde{\delta}^u_t)$ is to find out the expression of the right hand-side of eqq. (33) and solve the non-linear system for $(\tilde{\pi}_t, \tilde{\gamma}, \delta)$. Analogously to the 1-dimensional case, it shall be noted that the wealth process in the asset space is given by:

$$dx_i(t) = x_i(t)\left(\frac{1}{\delta^l_{t,i} + \delta^u_{t,i}}\right)\left[\left(4\pi_{t,i} - \frac{\psi_i}{2} + \mu_i \delta^u_{t,i}\right)dt + \delta^u_{t,i}\sigma^T dW_t\right] - \gamma_i \left(\frac{1}{\delta^l_{t,i} + \delta^u_{t,i}}\right)^2 x_i(t)dt, \tag{34}$$

which implies that the log-wealth process is given by:

$$
\begin{aligned}
d\log x_{t,i} = \frac{1}{x_{t,i}} dx_{t,i} - \frac{1}{2x_{t,i}^2} d < x_{t,i} >_t^2 = \\
\left(\frac{1}{\delta_{t,i}^l + \delta_{t,i}^u}\right) \left[4\pi_{t,i} - \frac{\psi_i}{2} + \left(\frac{\delta_{t,i}^u}{\delta_{t,i}^l + \delta_{t,i}^u}\right)\frac{\psi_i}{2} - \left(\gamma_i + \frac{\psi_i}{2}(\delta_{t,i}^u)^2\right)\left(\frac{1}{\delta_{t,i}^l + \delta_{t,i}^u}\right)\right] dt + \\
\left(\frac{\delta_{t,i}^u}{\delta_{t,i}^l + \delta_{t,i}^u}\right)\left[\left(\mu_i - \frac{1}{2}\psi_i\right) dt + \sigma_i^T dW_t\right] = \\
c_i(\pi_{t,i}) + \left(\frac{\delta_{t,i}^u}{\delta_{t,i}^l + \delta_{t,i}^u}\right) dX_t \quad \forall i \in \{1, 2, ...n\}.
\end{aligned} \tag{35}
$$

In vector form:

$$d\mathcal{X}_t = c(\pi_t) + \Delta_t^u dX_t. \tag{36}$$

Hence:

$$\mathbb{E}[p|\pi_t] = c(\pi_t) + \Delta_t^u \left(\mu - \frac{1}{2}\psi\right) dt, \tag{37}$$

$$Var[p|\pi_t] = (\Delta_t^u)^2 \Sigma dt. \tag{38}$$

Remembering eq. [8]

$$\tilde{\delta}_i^* = \frac{2\tilde{\gamma}_i + \mu_i^2 \tilde{\psi}_i^2}{4\tilde{\pi} - \frac{\tilde{\psi}_i^2}{2} + \tilde{\mu}_i\left(\tilde{\mu}_i - \frac{\tilde{\psi}_i^2}{2}\right)}, \tag{39}$$

and eqq. [9]:

$$
\begin{aligned}
\tilde{\delta}_i^u &= \frac{\tilde{\delta}_i^*}{2} + \tilde{\mu}_i, \\
\delta_i^l &= \frac{\delta_i^*}{2} - \mu_i.
\end{aligned} \tag{40}
$$

Plugging eqq. (39) and (40) into the left hand-side of eq. (33) and eqq. (9), (37) and (38) into the right hand-side of eq. (33), we get a system on n equations depending on $(\tilde{\pi}, \tilde{\gamma}, \delta)$. Hence, the optimal policy can be found by solving a system of non-linear equations imposing the following constraints, which guarantee the profitability of the solution in the asset space:

$$
\begin{aligned}
\delta_i \in (0, 4) \quad \forall i \in \{1, 2, ..., n\}, \\
\tilde{\gamma}_i > 0 \quad \forall i \in \{1, 2, ..., n\}.
\end{aligned} \tag{41}
$$

Finally, in order to choose a set $\{z_i(t)\}_{i=1}^n$ of optimal weights at time[3] t, the Agnostic Risk-Parity Portfolio can be adopted.

[3] Note that, given the current setting (constant μ and σ) such weights would be time-independent.

4. Agnostic Risk Parity Portfolio

At this point, we aim at treating each implementation of the optimal LP strategy discussed in sec. [2] as a synthetic asset in a portfolio optimization framework. In particular, we want to adjust the overall portfolio exposure of the LP in a way such that:

- The Optimal Liquidity Provision strategy is performed on each pool;
- The execution is coherent with a notion of optimal portfolio allocation to be chosen.

We evaluated a certain number of different methods (see Appendix A for a brief review). Typically, in any portfolio allocation problem, an investor aims at minimizing its risk under some constraint on expected returns, implying the need for the solution of a Markowitz(-like) optimal portfolio allocation problem. Some issues eventually arise in such cases:

- Usually, risk is measured by the mean of the standard deviation of the portfolio, depending on the covariance matrix. In such cases, the risk minimization approach leads to a concentration risk, arising from the fact that weights tend to be higher along the eigenvectors of the covariance matrix whose associated eigenvalues are lower.
- Parameter estimation uncertainty, arising from the intrinsic non-stationarity of financial time-series (at least unconditionally from the past state). It makes statistical estimators of expected returns and covariance matrices very uncertain.

The combination of both of the above issues often leads to poor out-of-sample performance of such portfolios.
To cope with the excessive concentration issue, Risk-Budgeting-constraint on asset holdings other that "underlying risk factor" holding have been proposed in literature (see Appendix A): if the first one actually doesn't diversify risk properly since it doesn't take into account asset correlations in the constraints, the second one usually leads to risk factors whose statistics differ a lot from the statistics of the individual assets, consequently implying portfolio allocations lacking of "financial soundness". Moreover, the difficulty in obtaining reliable estimations of the relevant parameters (i.e. expected values and covariance matrices) reverberate in the estimation of eigenvalues and eigenvectors, above all for smaller eigenvalues.
Those issues are efficiently dealt with by the Agnostic Risk Parity approach [1], briefly described hereafter.
The target is to obtain a method which:

1. Minimizes a certain distance between the risk factors in the "asset space" and the risk factors in the "spectral space";
2. Minimizes the bias due to uncertainty in parameter estimation, due to intrinsic non-stationarity of asset price and, consequently, return time-series.

Hence, let $r \in \mathbb{R}^n$ the standardized (i.e. demeaned and rescaled to unit variance) returns of each strategy over a time interval Δt and let $\Sigma^{std} \in \mathbb{M}_{\mathbb{R}}(n \times n)$ be their covariance matrix. Note that, since the returns are standardized, such matrix is actually a correlation matrix.
Moreover, let the vector of a (possibly biased) estimator of standardized expected returns $p \in \mathbb{R}^n$ arbitrarily chosen and let $C \in \mathbb{M}_{\mathbb{R}}(n \times n)$ an estimation of their covariance matrix, $\{\gamma_i\}_{i=1}^n \subset \mathbb{R}$ and $\{u\}_{i=1}^n \in \mathbb{R}^n$ being the related sequences of eigenvectors and eigenvalues, respectively.

As far as the first point is concerned, we aim at a representation of true returns and expected returns $\hat{r}$ and $\hat{p}$ respectively such that their Mahalanobis distance from their "asset space" counterpart r and p respectively is minimized, that is:

$$\underset{M \in \mathbb{M}_{\mathbb{R}}(n \times n):\, MM^T = I}{argmin} \quad \mathbb{E}\left[(\hat{a}(M) - r)^T K^{-1} (\hat{a}(M) - r)\right] = I, \tag{42}$$

with $\hat{a}(M) = M\Sigma_{std}^{-1} a$, $\forall a \in \{r, p\}$, $\forall K \in \{\Sigma^{std}, C\}$.

In [1] (Appendix A), it is shown that the solution to the aforementioned problem is given by the identity matrix $I \in \mathbb{R}(n \times n)$. Accordingly, the following rotations of r and p are performed:

$$\hat{r} = \Sigma_{std}^{-1/2} r, \tag{43}$$

where:

$$\Sigma_{std}^{-1/2} := \sum_{i=1}^{n} \frac{1}{\sqrt{\lambda_i^{std}}} v_i^{std} (v_i^{std})^T. \tag{44}$$

$\{\lambda_i^{std}\}_{i=1}^n$ with $\{v_i^{std}\}_{i=1}^n \subset \mathbb{R}^n$ components of the matrix $V^{std} \in \mathbb{M}_{\mathbb{R}}(n \times n)$ of the (orthogonal) eigenvectors of Σ^{std}.

Analogously, for vector p:

$$\hat{p} = C^{-1/2} p, \tag{45}$$

$$C^{-1/2} := \sum_{i=1}^{n} \frac{1}{\sqrt{\gamma_i}} h_i h_i^T, \tag{46}$$

with $\{h_i\}_{i=1}^n \subset \mathbb{R}^n$ components of the matrix $H \in \mathbb{M}_{\mathbb{R}}(n \times n)$ of the (orthogonal) eigenvectors of C.
Now, define the portfolio:

$$g := \omega < \hat{r}, \hat{p} >= \omega \hat{r}^T \hat{p} =< \hat{\pi}, \hat{r} >, \tag{47}$$

where $\omega \in \mathbb{R}^+$ is a suitable normalization factor and $\hat{\pi} := \omega \hat{p}$ are the portfolio weights.
Now, let $g_i := \pi_i \hat{r}_i = \omega \hat{p}_i \hat{r}_i$. This portfolio, depending on the expected returns of the underlying risk factors $\hat{r}$, has the following desirable features:

- $\mathbb{E}[g_i^2] = 1 \quad \forall i, j \in \{1, 2, ..., n\} : i \neq j \quad \forall i, j \in \{1, 2, ..., n\} : i = j$ (unitary risk associated to each underlying risk factor);
- $\mathbb{E}[g_i g_j] = 0 \quad \forall i, j \in \{1, 2, ..., n\} : i \neq j$ (uncorrelation);
- $g_R :=< \mathcal{R}(r), \mathcal{R}(p) >= r^T R^T R p = r^T p$ (rotation invariance).

Where $R \in \mathbb{M}_{\mathbb{R}}(n \times n)$ is the rotation matrix representing the rotation operator $\mathcal{R}$.
In particular, unitary risk associated to each risk factor and null correlation define the scale invariance property, which is important in order to let all the assets be comparable, no matter the scale of each one. Moreover, the rotation invariance implies that rotation leads to the same portfolio expected return, no matter .
By substituting eq. (45) and (43) into eq. (47) we obtain:

$$g = \omega < C^{-1/2}p, \Sigma_{std}^{-1/2}r >= \omega p^T \left(C^{-1/2}\right)^T \Sigma_{std}^{-1/2} r = \pi^T r =< \pi, r >, \tag{48}$$

where:

$$\pi := \omega p^T \left(C^{-1/2}\right)^T \Sigma_{std}^{-1/2} = \omega (\Sigma_{std}^{-1/2})^T C^{-1/2} p. \tag{49}$$

Such portfolio is defined *Eigenrisk-Parity Portfolio*. In facts, let the second order moment of π:

$$Var[\pi \cdot \sqrt{\gamma_i} v_i^{std}] = \mathbb{E}[(\pi \cdot v_i^{std})^2]\gamma_i = \omega^2 \left(\frac{1}{\sqrt{\gamma_i}}\right)^2 \mathbb{E}[(v_i^{std} \cdot C^{-1/2}p)^2]\gamma_i = \omega^2 \quad \forall i \in \{1, 2, ..., n\}. \tag{50}$$

That is, the variability is the same for all directions (eigenvectors) in which risks spread out. Moreover, by avoiding to target minimum variance, it effectively tackles the issue of over-allocation to small eigenmodes, whose estimation is usually quite unstable. It rather aims at an "Occam razor" allocation, i.e. relying on a minimum information set and (implicit or explicit) assumptions about the data generating process of both realized asset returns and expected asset returns. In particular, it is referred to as *Agnostic Risk-Parity portfolio* in case no particular relationship is supposed among expected return estimators, i.e. in case $C := I$. Hence, the Agnostic Risk-Parity portfolio is given by:

$$\pi_{ARP} := \omega \Sigma_{std}^{-1/2} p. \tag{51}$$

Instead, in case $C := \Sigma_{std}$ and $\omega = \frac{1}{\mathbf{1}^T \Sigma_{std}^{-1} w}$ we recover the Markowitz optimal portfolio (54). In order obtain an unbiased estimation of the covariance matrix, the authors of [1] suggest to use the Rotation Invariant Estimator introduced in [2]: as the name suggests, it is invariant under (random) variations of the eigen-directions of the rotation represented by the eigenvector matrix of the covariance matrix.

5. The Strategy

Hence, a possible algorithm follows:

- At time $t_0 = 0$:
  1. Estimate the vector of means and the covariance matrix. For an efficient estimator of the covariance matrix one can resort to the Rotational Invariant Estimator (see [[2]]).
  2. Recast the optimal allocation problem by representing the log-returns of the spot exchange rates X_t in the "underlying risk factor" space as done in eq. (18) and obtain the wealth process in eq. (22).
  3. Select a rebalancing frequency Δt and, hence, a time-grid $\{t_i\}_{i=1}^n$ such that $t_k = k\Delta t \quad \forall k \in \{1, 2, ..., n\}$.
- At each time-step t_n:
  1. Find out the optimal strategy as prescribed by [3] for each (independent) underlying risk factor.
  2. Solve the non-linear system of eq. (33) and find out the optimal policy δ^* to be executed in each real pool.

3. Update the vector of expected returns and rebalance the exposures to each real liquidity pool according to the Agnostic Risk Parity portfolio weights.
4. Execute the optimal liquidity provision strategy in each real liquidity pool.

6. Conclusions

In this paper we dealt with optimal liquidity provision strategies in crypto-currency Decentralized Exchanges (DEXs) working on the UniswapV3 protocol. We outlined the optimal liquidity strategy proposed in [3] and we assessed its performance both in a simulated WETH-USDC pool and in a backtesting framework. We found out that the strategy did not perform better than some simpler liquidity provision strategy, namely the spot-tracking LP strategy and, in the backtest case, the static LP strategy. Hence, we went on proposing an extension to a multi-pool setting: the liquidity provider wants to allocate its capital optimally across more than one exchange. Exploiting the spectral properties of the Wiener process, we recovered the possibility to use the optimal single-pool strategy according some optimality conditions about the overall capital allocation. The implementation and assessment of the performance will be carried out in a future work.

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A. Appendix

A.1 Mean-Variance Portfolio Optimization

The most famous portfolio optimization framework was proposed, in 1952, by Harry Markowitz [6], earning him the Nobel Price in Economics in 1990: the so-called Mean-Variance Optimal Portfolio is supposed to be the one that minimizes the overall portfolio variance under a minimum expected return requirement $\mu^* \in \mathbb{R}$ and the so-called full-invested constraint, requiring that all the capital is invested in the available universe of assets. In formulas:

$$\begin{aligned} \min_w \quad & Var_{ptf}(w) := w^T \Sigma w \\ & \text{s.t.} \\ & w^T \mu \;\; \geq \;\; \mu^* \\ & w^T \mathbf{1} \;\; = \;\; 1. \end{aligned} \tag{52}$$

Equivalently, the problem can be reformulated in terms of maximum Sharpe Ratio (SR) under full-invested constraint:

$$\begin{aligned} \max_w \quad & SR_{ptf}(w) := \frac{w^T \mu}{\sqrt{w^T \Sigma w}} \\ & \text{s.t.} \\ & w^T \mathbf{1} \;\; = \;\; 1. \end{aligned} \tag{53}$$

The solution exists in closed form and is given by:

$$w^*_{\text{MVO}} = \frac{\Sigma^{-1} \mu}{\mathbf{1}^T \Sigma^{-1} \mu}. \tag{54}$$

In case of absence of minimum expected return constraint in (52), the Markowitz optimal portfolio is referred to as the Global Minimum Variance (GMV) portfolio. The solution is given by:

$$w^*_{\text{GMV}} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}. \tag{55}$$

Portfolios of Markowitz-type (either MSR or GMV) resort to a principle of minimum variance. Hence, they lead to portfolios which are very concentrated in the lowest variance asset. This is even more evident if we recast the problem in the underlying risk factor space, that is, formulating the problem with respect to the principal components of the covariance matrix other than the physical asset space. By substituting eq. (15) into the objective function in eq. (52) and defining $w := Vx$ we obtain:

$$\begin{aligned} \min_w \quad & Var_{ptf}(w) := w^T \Lambda w \\ & \text{s.t.} \\ & w^T \mu \;\; \geq \;\; \mu^* \\ & w^T \mathbf{1} \;\; = \;\; 1. \end{aligned} \tag{56}$$

In order to attain the minimum possible variance, it is clear that the weights will be higher for the principal component with lower eigenvalues, exposing the portfolio to a material concentration risk. Indeed, this would be optimal under the assumption that the investor is able to estimate accurately the covariance matrix Σ and the expected return vector μ. However, this is usually very difficult when dealing with returns of financial assets, often suffering from lack of stationarity or even ergodicity. Hence, the lack of reliable estimators

for covariance matrices and expected returns causes the over-allocation to a lower-variance asset a drawback of Markowitz-type optimal portfolios.
In order to deal with such drawbacks, a certain number of approaches have been proposed in literature: most of them, aim at sorting out the concentration problem by increasing the level of diversification of optimal portfolios by imposing so-called risk budget constraints and/or exploiting rotation invariance properties of elliptical probability distributions (e.g. recasting the problem in some equivalent space characterized by the spectral properties of estimated covariance matrices, from now on called underlying risk factor space).
However, the latter fixes miss to deal with the main issue of Markowitz-type optimal portfolios, that is, the intrinsic uncertainty in the estimation of expected returns and covariances. A possible solution to this issue, along with some other drawbacks arising in the latter alternative approaches, is proposed in [1] which will be described in the last section.

A.2 Risk-Budgeting and Risk-Parity

A suitable approach to increase the diversification of a Markowitz-type optimal portfolio is the risk-budgeting approach, which consists in introducing a set of nonlinear constraints to the Markowitz optimal portfolio such that the overall quote of variance of the portfolio that should be allocated to a single asset in eq. (52) is fixed:

$$\begin{array}{rcl} \min\limits_{w} & Var_{ptf}(w) := w^T \Sigma w & \\ & \text{s.t.} & \\ w^T\mu & \geq & \mu^* \\ \frac{\Sigma w}{w^T \Sigma w} \times w & = & b \\ w^T \mathbf{1} & = & 1, \end{array} \tag{57}$$

where the entries of $b \in [0,1]^n$ represent the maximum portfolio variance fraction allowed to the i-th un derlying risk factor (see [8], page 80 for the proof). In particular, when $b := \mathbf{1} \cdot \frac{1}{n}$, the risk-budgeting constraint is referred to in a self-explicable way: *risk-parity* constraint. Such a problem is convex: although no closed-form solution is available, it is solvable by mean of any numerical algorithm.
In particular, removing the minimum expected return required constraint and setting the risk-parity constraint, lead to the following modification of the Global Minimum Variance optimal portfolio problem (55):

$$\begin{array}{rcl} \min\limits_{w} & Var_{ptf}(w) := w^T \Sigma w & \\ & \text{s.t.} & \\ \frac{\Sigma w}{w^T \Sigma w} \times w & = & \mathbf{1} \cdot \frac{1}{n} \\ w^T \mathbf{1} & = & 1. \end{array} \tag{58}$$

However, the optimal solutions to the problem just introduced are specified in the asset domain, that is, the optimal weights are set such that they satisfy a constraint expressed in terms of variance of the single asset, without taking into account that correlations among asset make drawdowns, due to a shock in the returns of one asset, propagate also to other assets, hence not taking into account the underlying risk factors potentially driving more than asset.
In order to cope with this issue, the same problem can be specified in the underlying risk factor domain, analogously to what was done in (58) by imposing risk-parity constraints with respect to underlying risk factors:

$$
\begin{aligned}
\min_{w} \quad & Var_{ptf}(\hat{w}) := \hat{w}^T \Lambda \hat{w} \\
\text{s.t.} & \\
\frac{\Lambda \hat{w}}{\hat{w}^T \Lambda \hat{w}} \times \hat{w} \quad &= \quad \mathbf{1} \cdot \frac{1}{n} \\
\hat{w}^T \mathbf{1} \quad &= \quad 1.
\end{aligned} \tag{59}
$$

A.3 Max-Entropy

The max-entropy approach tackles the problem of risk concentration differently from the risk-budgeting approach: the diversification requirement is expressed by looking for the set of weights of the different underlying risk factors that maximize an entropic measure.
First of all, we introduce the following entropic measures, known as Rényi entropies, to be used as measures of portfolio diversification:

$$
\mathrm{E}_\alpha(w) = ||w||_\alpha^{\frac{\alpha}{1-\alpha}} = \left(\sum_{k=1}^N w_k^\alpha \right)^{\frac{1}{1-\alpha}}, \qquad \alpha \geq 0, \quad \alpha \neq 1. \tag{60}
$$

In particular, for $\alpha \to 1$, we recover the Shannon Entropy:

$$
\mathrm{E}_1(w) = \exp\left(-\sum_{k=1}^N w_k \ln(w_k) \right), \tag{61}
$$

whereas, for $\alpha = 2$:

$$
\mathrm{E}_2(w) = \frac{1}{\sum_{i=1}^n w_k^2}. \tag{62}
$$

Such measures vary from 1 to n depending on whether the portfolio is fully invested in one asset (i.e. $\exists k \in \{1, 2, ..., n\} : w_k = 1;\ w_j = 0 \quad \forall j \in \{1, 2, ..., n\} : j \neq k$) or in all assets in the same measure (i.e. $w_k = \frac{1}{n}\ \forall k \in \{1, 2, ..., n\}$).
Then, let:

$$
p(\hat{w}) := \Lambda \hat{w} \times \hat{w}. \tag{63}
$$

Then, the optimum portfolio weights $\hat{w} \in [0, 1]^n$ are the weights solving the following maximization problem:

$$
\begin{aligned}
\max_{\hat{w}} \quad & E_\alpha(\hat{w}) \\
\text{s.t.} & \\
\hat{w}^T \mathbf{1} \quad &= \quad 1.
\end{aligned} \tag{64}
$$

It can be shown that the max-entropy approach for $\alpha = 2$ is equivalent to impose a shrinkage to the covariance matrix in the Risk-Parity Global Minimum Variance optimal portfolio problem (see Proposition 4 of [4]).

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