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Risks Aggregation, Tail Dependence and Beyond

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September 2025

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Research Paper Series

Year 2025 - Issue Number 78

Last published issues are available online:
http://www.iasonltd.com/research

Front Cover: **Atanasio Soldati**, *Untitled*, 1952.

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Executive Summary

In an era marked by heightened uncertainty and intricate interconnections across financial systems, robust risk aggregation and dependence modelling have become critical pillars. Within the European insurance and banking sectors, capturing inter-risk dependencies, especially in the tails of distributions, is fundamental for assessing systemic vulnerabilities and ensuring solvency. This paper opens with an overview of the historical and regulatory evolution of risk aggregation practices in the financial sector, with a specific focus on insurance. Particular attention is given to the development of the Solvency II framework and the supervisory role of EIOPA, laying the groundwork for a more in-depth examination of dependence structures and the statistical tools used to model joint risk behaviour. A key focus is on tail dependence, a concept of paramount importance in the context of extreme loss events and capital adequacy. The discussion combines theoretical foundations with empirical illustrations to highlight how failing to properly account for tail dependence can lead to significant underestimation of aggregate exposures. The final section provides the methodologies to conduct a comparative impact analysis under the Solvency II capital requirements regime. By integrating regulatory context with quantitative rigour, the paper contributes to ongoing efforts to improve capital adequacy, model validation, and stress testing practices. This contribution is particularly relevant today, as both the banking and insurance industries face growing pressure to adopt models that better reflect real-world joint risk behaviour.

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This article was written in collaboration with Zineb Karfa who at the time was working for Iason Consulting.

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Risks Aggregation, Tail Dependence and Beyond

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In recent years, the stability of financial systems has been increasingly challenged by global shocks, heightened complexity, and growing interconnections across institutions and markets. These dynamics have elevated the importance of accurately modelling aggregated risk and capturing its dependencies, particularly in the tails of loss distributions, as a foundational element of both regulatory supervision and internal risk management. This is especially relevant in the insurance and banking sectors, where low-probability, high-severity events can compromise solvency and propagate systemic risk. In response to these challenges, regulators have responded with comprehensive frameworks aimed at ensuring resilience. The Solvency II directive, governing the European insurance industry, and the Basel III framework, guiding banking regulation, both emphasise the importance of solid capital requirements and sound risk aggregation practices. Central to these efforts is the challenge of capturing inter-risk dependence, particularly in the tails of loss distributions, where traditional correlation-based approaches often fall short.
This paper explores the historical and regulatory development of risk aggregation, with a specific emphasis on the insurance sector under Solvency II and the supervisory role of EIOPA. It proceeds to formalise a range of statistical dependence measures and to introduce various risk aggregation methodologies, emphasizing the limitations of linear correlation in favour of more reliable representations. A detailed treatment of tail dependence metrics follows, highlighting their significance in capturing extreme co-movements and informing stress testing and scenario analysis. The final section evaluates how alternative dependence structures and aggregation methods influence capital requirements under Solvency II and Basel III. This provides a framework for enhancing model validation practices and strengthening the resilience of financial institutions in the face of tail events and systemic shocks.

1. Risk Aggregation: Regulatory Evolution

The aggregation of risks has become a cornerstone of modern financial regulation, especially as financial systems grow more complex and interconnected. This chapter traces the evolution of regulatory approaches to risk aggregation, examining how major frameworks like Basel and Solvency have adapted to better capture dependencies and systemic vulnerabilities.

1.1 Evolution of Risk Aggregation and Regulatory Motivation

The evolution of regulatory frameworks for risk aggregation in the banking and insurance sectors has been driven by successive financial crises and the growing complexity of financial systems.
The European Union (EU) has progressively enhanced its regulatory framework to address systemic vulnerabilities, emphasizing the importance of understanding extreme co-movements in financial markets. Tail dependence, which captures the propensity of asset returns to exhibit extreme co-movements, poses significant challenges to traditional risk management approaches that often assume normality and independence.
In response to these challenges, European regulatory bodies have implemented measures to bolster financial stability. The European Central Bank (ECB), in its *Financial Stability Review - November*

2024 [23], highlights the increasing interconnectedness of financial institutions and the potential for contagion during periods of market stress. Moreover, the adoption of advanced risk assessment tools, such as copula-based models, has been encouraged to better capture the complexities of joint extreme events.

Two dominant frameworks, the Basel Accords for banking and the Solvency frameworks for insurance, have emerged as the cornerstones of global financial regulation. Though initially distinct, these regimes have progressively converged in their recognition of systemic risk, the need for sophisticated modelling techniques, and the integration of forward-looking stress scenarios.

Figures 1 and 2 present the key regulatory milestones in the Basel and Solvency frameworks, respectively. The timelines highlight the parallel evolution of banking and insurance regulation, from foundational directives to the most recent reforms, illustrating how both regimes have responded to financial crises and advanced toward risk-based capital standards.

The regulation of financial institutions has traditionally relied on models that assume linear correlations, normal distributions, and relative independence of risk factors. However, as financial markets became more interconnected and volatile, regulators began to acknowledge the shortcomings of these assumptions, particularly during crises, when asset returns tend to move together in extreme and non-linear ways. This phenomenon, known as tail dependence, reflects the heightened probability of joint extreme losses across institutions, asset classes, or geographies. Understanding and managing aggregated risks in this context has thus become essential to the stability of both banks and insurers.

To better understand the trajectory of regulatory development, we begin with the banking sector's response to risk aggregation challenges under the Basel framework.

1.2 Regulatory Frameworks in Banking: The Basel Accords

International Convergence of Capital Measurement and Capital Standards (1988) [6], also known as Basel I Accord, marked the first international attempt to standardise capital requirements, introducing a simple, additive system based on fixed risk weights. However, this model assumed linear relationships between exposures and failed to recognise correlations or tail events. As a result, it underestimated systemic vulnerabilities, particularly in times of financial stress.

In response to these shortcomings, *Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework* (2004) then revised in *International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version* (2006) [7] introduced a more nuanced and risk-sensitive approach. It leveraged internal ratings-based models that account for Probability of Default, Loss Given Default, and Exposure at Default. Although this marked significant progress, it did not directly address tail dependence. Instead, it relied on Gaussian copulas and normal distributions in portfolio models, which proved inadequate in capturing extreme joint losses, as revealed during the 2008 financial crisis.

Building on these developments and considering the global financial crisis *Revisions to the Basel II market risk framework* (2011) [8], known as Basel 2.5, introduced the Stressed Value-at-Risk (VaR) and the Incremental Risk Charge. *Basel III: A global regulatory framework for more resilient banks and banking systems* (2011) [4], which came into effect after 2013 with the implementation of the *Capital Requirements Regulation (EU) No 2013/575* [28], represented a substantial enhancement to regulatory rigor. It introduced the Liquidity Coverage Ratio, Net Stable Funding Ratio, capital conservation buffers, and leverage ratios. Furthermore, Basel III began a transition from VaR to Expected Shortfall (ES) under the *Fundamental review of the trading book: A revised market risk framework* [5], thereby improving the framework's ability to capture tail risk. Although tail dependence was not formally embedded, Basel III's stress testing frameworks simulate co-movements under systemic stress and promote more realistic modelling of non-linear dependencies.

1.3 Regulatory Frameworks in Insurance: The Solvency Regimes

In parallel with the evolution of banking regulations, the insurance sector also underwent a transformation through the Solvency frameworks.

Directive 2002/13/EC [29] regarding the solvency margin requirements for non-life insurance undertakings and *Directive 2002/83/EC* [30] concerning life assurance, introduced Solvency I. This

framework was based on simplistic, factor-based formulas and assumed independence between risk types, ignoring potential diversification and systemic aggregation effects, particularly during market disruptions.
Directive 2015/35/EU [26], known as Solvency II, replaced this static regime with a more dynamic and risk-based system. It introduced the Solvency Capital Requirement (SCR), calculated using either a standard formula or internal models, to capture risk exposures over a one-year horizon with 99.5% confidence. Crucially, Solvency II allowed the use of copula functions, especially t-copulas and Clayton copulas, to model asymmetric and nonlinear dependencies between risks. In addition, it mandated regular stress testing and scenario analysis, extending to climate and systemic shocks, and provided mechanisms for group-level aggregation and oversight.

1.4 Methodological Advances and Future Directions

To effectively capture aggregated risks, both Basel III and Solvency II allow banks and insurance companies to use internal models, supported by advanced statistical techniques and solid governance frameworks.
These internal models are subject to supervisory approval and rigorous validation procedures under both Solvency II and Basel III frameworks. This includes requirements to demonstrate the models' reliability, appropriate use within risk management practices, and compliance with quantitative and qualitative standards set by regulators.
Copula-based modelling and heavy-tailed distributions are among the techniques now commonly used to quantify joint extreme losses across risk types and asset classes. Model validation includes benchmarking, back-testing and supervisory audits. Where empirical data are insufficient, Basel III and Solvency II allow expert judgment-provided it is transparent and justifiable.
Complementing these regulatory efforts, academic and actuarial communities have made significant contributions to the improvement of risk aggregation methods.
Researchers such as Bernard and Vanduffel (2016) [9] have argued that correlation-based models tend to overstate diversification benefits when tail dependence is present. Similarly, Bruneton (2011) [10] criticised Gaussian copulas for their inability to accurately capture extreme co-movements. In response to these limitations, Marri and Moutanabbir (2021) [41] proposed the use of Generalised Archimedean Copulas to enhance the precision of capital aggregation modelling.
Despite methodological advances, the adoption of internal models remains uneven across jurisdictions and sectors. In the EU, larger financial institutions, especially those classified as Group 1, are more likely to use internal models, given their greater resources and more complex risk profiles. In contrast, smaller entities typically rely on standard formulas due to the high costs and regulatory demands of model development and approval. This pattern is particularly evident in the insurance sector, where, as of 2020, only 14 insurers in Italy had received full or partial internal model approval [34]. Among banks, internal model use is also more common among larger institutions, although regulatory developments have increased scrutiny and promoted broader use of standardised approaches [20].
Although the Basel and Solvency frameworks serve different sectors, they are increasingly converging in their recognition of systemic risk, contagion, and tail dependence. The aftermath of the global financial crisis, and more recently the COVID-19 pandemic and a series of geopolitical shocks, have reinforced the need for regulatory models that move beyond average-case scenarios and address the complexities of extreme events.
Today, both frameworks are evolving to incorporate advanced methodologies that reflect these concerns. There is a growing emphasis on the use of copula models and heavy-tailed distributions within internal risk assessments, allowing institutions to more accurately capture dependencies and extreme co-movements among risk factors. In parallel, stress testing frameworks have become more sophisticated, aiming to model joint extreme losses across various business lines and risk types, thereby providing a more holistic view of institutional vulnerability.
Moreover, scenario aggregation techniques are being refined to better reflect the non-linearities in financial markets. These improvements help regulators and institutions alike to understand how risks can interact and amplify under adverse conditions, enhancing the reliability of risk management practices.
Looking ahead, the integration of emerging threats such as climate risks, cyber risks, and geopolitical

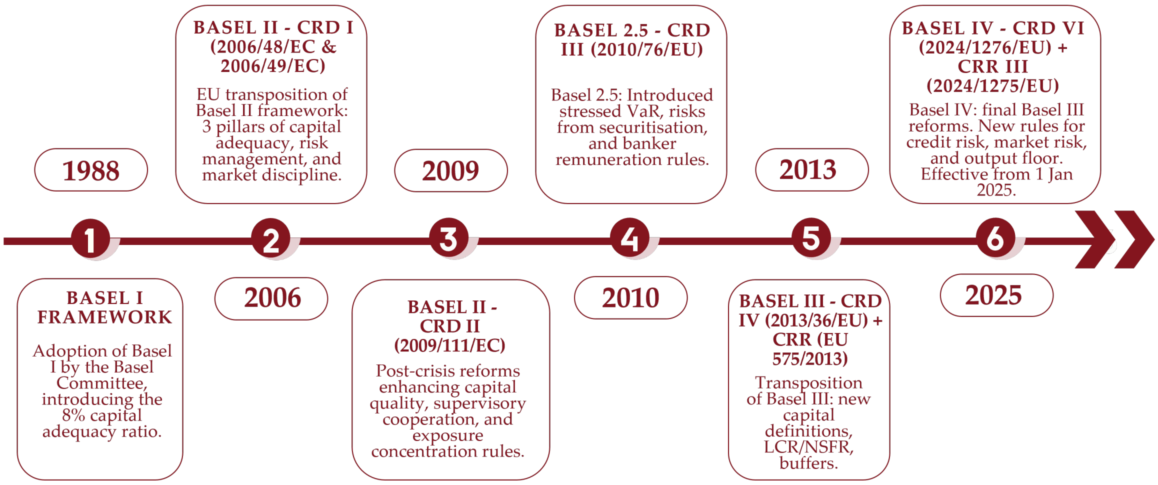


FIGURE 1: Evolution Timeline of Basel Regulatory Framework

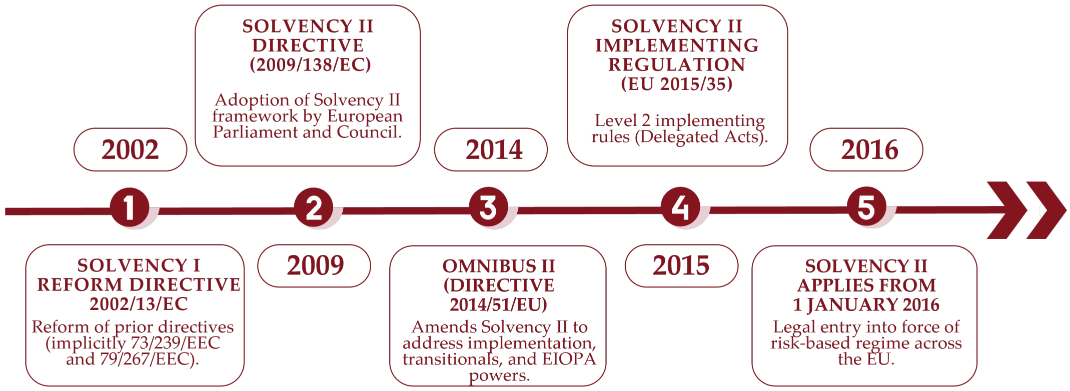


FIGURE 2: Evolution Timeline of Solvency Regulatory Framework

instability is expected to further influence regulatory models. This evolution will likely heighten the focus on non-linear tail dependencies and network-based contagion models, ultimately pushing both Basel and Solvency frameworks toward more dynamic, system-wide approaches to risk aggregation.
Environmental, Social, and Governance (ESG) factors are becoming increasingly central to regulatory policy and financial risk management. The United Nations Environment Programme Finance Initiative (UNEP FI) [52] has issued specific guidance on integrating ESG risks into insurance underwriting practices. Empirical evidence from Allianz Commercial [3] supports the predictive power of ESG metrics in assessing insurance risks and designing risk frameworks. Concurrently, artificial intelligence and machine learning (AI/ML) are reshaping internal modelling practices, introducing both new analytical capabilities and challenges related to governance and explainability. SupTech-technology-enabled supervision is being advanced by institutions like the ECB and the Cambridge SupTech Lab to enhance regulatory responsiveness and precision.
Another persistent challenge is model uncertainty, particularly under varying dependency structures. Embrechts et al. [19] introduced the concept of aggregation robustness to address this issue, while Cambou and Filipovic (2017) [12] provides further insights into improving the resilience of capital models under diverse assumptions.

1.5 Global Comparison

When comparing international approaches, it becomes clear that regulatory philosophies and capabilities vary widely. The following examples focus on the insurance sector, where differences in regulatory philosophy, model usage, and risk aggregation practices are particularly pronounced across jurisdictions.

Country/Region	Regulatory Framework	Supervisory Authority
European Union	Solvency II	European Insurance and Occupational Pensions Authority (EIOPA)
Switzerland	Swiss Solvency Test (SST)	Swiss Financial Market Supervisory Authority (FINMA)
United States	NAIC Risk-Based Capital (RBC)	National Association of Insurance Commissioners (NAIC)
Japan	Solvency Margin Ratio (moving to EV-based regime in 2025)	Financial Services Agency (FSA)
Singapore	RBC 2	Monetary Authority of Singapore (MAS)
China	C-ROSS / C-ROSS II	China Banking and Insurance Regulatory Commission (CBIRC)
Canada	LICAT (formerly MCCSR for life, CCIR for P&C)	Office of the Superintendent of Financial Institutions (OSFI)

TABLE 1: Regulations and authorities in the World

As depicted in Table 1, the EU and Switzerland have embraced model-intensive, principle-based frameworks that formally recognise tail dependence and allow for group-level aggregation. In contrast, the United States' NAIC Risk-Based Capital (RBC) framework relies on rule-based formulas with limited flexibility for internal models or inter-entity diversification. In Asia, regulatory maturity is mixed: while Japan and Singapore move closer to Solvency II, China's C-ROSS system reflects early steps toward internal model alignment. Canada's MCCSR framework, meanwhile, limits inter-risk diversification and enforces capital additivity across risk types.
Switzerland's Swiss Solvency Test (SST) stands out for its balance of flexibility and oversight. It allows both standard and internal model use and supports holistic diversification through cluster modelling and intra-group transactions subject to regulatory approval.
In conclusion, the future of risk aggregation regulation lies in continued adaptation and convergence. As global financial systems become more interconnected and exposed to complex systemic risks, from climate change and cyber threats to geopolitical instability, regulatory frameworks must evolve to embrace data-driven modelling, stronger governance mechanisms, and coordinated global oversight.

2. Risk Aggregation: Supervisory Standards

The practical implementation of dependence modelling and risk aggregation in the insurance and banking sectors rests on a sophisticated regulatory framework. Unlike the historical and structural perspectives outlined previously, this chapter identifies and discusses the most influential and current regulatory and technical sources that govern how institutions are expected to manage, validate, and disclose their aggregation logic. These sources include legal instruments, supervisory guidelines, technical papers and professional standards, which together form the operational scaffolding of modern solvency and risk capital regimes.

2.1 Regulatory Instruments and Supervisory Guidelines in Insurance

A foundational reference for the insurance sector is *Commission Delegated Regulation (EU) 2015/35* [26], which supplements Solvency II and establishes the correlation structure within the standard formula. Regulation prescribes correlation coefficients across risk modules and presumes a linear dependency framework that determines diversification benefits when aggregating market, credit, life, health, and non-life risks. This linear correlation method offers computational simplicity and ease of supervisory comparison but has been widely criticised for failing to capture the nonlinear and tail-dependent relationships observed during stress events. Alternative techniques, such as copula-based models, have been proposed in literature to address these shortcomings, although they introduce additional complexity and model risk. IVASS, Italy's insurance regulator, echoes and operationalises these Solvency II principles through its *Regolamento n. 32 IVASS del 9 novembre 2016* [36], which requires Italian insurers to demonstrate that their internal aggregation methods are comprehensive and forward-looking. The regulation emphasises that ORSA processes should explicitly incorporate inter-risk dependencies, including under adverse scenarios, and that these dependencies be regularly reviewed as part of the risk governance process.

In addition, *IVASS Regolamento n. 20/2016* [35] allows the use of independent experts to evaluate internal models, especially in case of complex aggregation methodologies that cannot be validated using standard tools. These external professionals, often drawn from actuarial, quantitative finance or audit backgrounds, provide technical assurance to IVASS on the soundness of models dealing with stochastic dependencies, copula calibration, or tail aggregation methods. This complements broader European-level initiatives led by EIOPA, including the *Guidelines on ORSA (EIOPA-BoS-14/259)* [24], which provide supervisory expectations for documenting and validating internal aggregation logics. *EIOPA's 2020 Study on Diversification in Internal Models* [25] offers further insights. The report investigates how insurers across Europe quantify diversification, noting wide variability in the implementation of copulas and correlation matrices. The study highlights that even within a harmonised framework like Solvency II, substantial discretion exists in calibrating dependency structures, which may result in materially different capital outcomes. Such discretion underscores the importance of model governance and supervisory benchmarking.

2.2 Regulatory Instruments and Supervisory Guidelines in Banking

For banking institutions, the *Capital Requirements Regulation (CRR, Regulation EU No. 575/2013)* [28] and the *Capital Requirements Directive (CRD IV, Directive 2013/36/EU)* [27] transpose the Basel III framework into EU law. While the CRR defines Pillar 1 capital requirements using separate formulas for credit, market and operational risk, it does not allow cross-risk diversification. Aggregation, therefore, becomes essential under Pillar 2 via the Internal Capital Adequacy Assessment Process (ICAAP), as mandated by *CRD IV*. Supervisory authorities use the SREP (Supervisory Review and Evaluation Process) to assess how banks aggregate risks, evaluate diversification assumptions and determine if additional capital buffers are required. In practice, some supervisors have imposed Pillar 2 Requirements (P2R) or Pillar 2 Guidance (P2G) to counteract what they view as unjustified capital relief resulting from optimistic internal aggregation. The European Banking Authority (EBA) further reinforces this framework through its *2016 Guidelines on ICAAP and ILAAP* [22], which obligate banks to justify their risk aggregation techniques and quantify any diversification effects claimed. Banks must demonstrate through data, stress testing and scenario analysis that their dependencies are realistic and consistent with past risk behaviour.
On a global level, the Basel Committee's 2017 finalisation of Basel III introduced an output floor that limits the benefits derived from internal model variability, thereby indirectly constraining the effects of optimistic dependency modelling. While the final floor is set at 72.5% of standardised capital requirements, it is being phased in from 50% in 2025, gradually increasing to 72.5% by 2030. This imposes a ceiling on the extent to which model-derived diversification can reduce capital and aims to ensure comparability across institutions. In the insurance domain, the *IAIS's Insurance Core Principle 16* [33], stipulates that internal models used for solvency assessments must integrate all material risks and dependencies, and their assumptions must be transparent, documented, and validated. Supervisors are encouraged to assess not only the quantitative integrity of dependency modelling but also the appropriateness of its use within enterprise risk management frameworks.

2.3 Divergent Approaches to Cross-Risk Diversification in Banking and Insurance

One of the most fundamental distinctions between the banking and insurance regulatory frameworks lies in their treatment of cross-risk diversification in capital requirement calculations. While both sectors aim to ensure financial resilience, their respective regulatory philosophies diverge sharply in terms of how risk aggregation is handled.
The banking sector, under the Basel III framework, particularly within Pillar I, takes a notably conservative stance. Regulatory capital requirements are calculated for credit, market, and operational risks separately, and these are simply added together without allowance for any diversification effects between them. Even internal models are developed and validated separately, without an integrated cross-risk modelling approach. By treating each risk as independent in the regulatory capital calculation, Basel reinforces a preference for transparency, comparability, and operational robustness.
By contrast, Solvency II explicitly allows for diversification across risk modules. Recognising that insurance portfolios often involve a broad spectrum of interconnected risks, from market and credit

to life, health, and non-life, the Solvency II standard formula explicitly incorporates a diversification framework.
Although Pillar I of the Basel framework disallows diversification effects, Pillar II, through the Internal Capital Adequacy Assessment Process, permits banks to adopt more nuanced aggregation approaches. However, the embrace of diversification is conditional and often heavily constrained. Supervisors tend to be cautious, demanding strong evidence of diversification effects, especially under stress scenarios as emphasised by the European Banking Authority, in its *Supervisory Review and Evaluation Process Guideline* (2022) [21].
The differing treatment of diversification reflects fundamental differences in sectoral risk characteristics. The Basel framework prioritises comparability and avoids complex modelling of heterogeneous risk interactions, favouring simplicity and robustness. Solvency II, having developed later, takes a more integrated and calibrated approach to the insurer's portfolio of risks, recognising inter-risk diversification both in its standard formula and in internal model approvals.

2.4 Standards, Academic Contributions and Emerging Challenges

Regulatory discourse has also been enriched by professional standards and academic contributions. The Actuarial Standards Board in its *Actuarial Standard of Practice No. 55: Capital Adequacy Assessment*. [2] provides actuarial guidance on evaluating capital adequacy and explicitly instructs actuaries to consider risk correlations and dependencies in their solvency assessments. The standard further requires disclosure of the methods, data sources, and limitations used in dependency modelling, thereby enhancing the transparency of actuarial evaluations. Academic studies, such as those by Bernard and Vanduffel (2016) [9], critique existing models for neglecting tail dependence and advocate for multivariate frameworks using copulas. These authors also propose techniques to quantify the model risk inherent in selecting a particular dependency structure.
Together, these regulatory instruments, supervisory expectations and academic insights establish the baseline against which institutions must design and validate their risk aggregation methodologies. They also offer a comparative lens through which to evaluate the interplay between regulatory conservatism and modelling sophistication. The challenge for firms is to balance compliance with the prudential need for resilience, while preserving the economic benefits of legitimate diversification. This ongoing tension continues to shape the evolution of supervisory practices and the future direction of capital regulation.

3. Measures of Dependency

Before delving into specific measures, it is essential to understand why dependency matters and how it influences the overall risk profile of a financial institution. This chapter explores various tools used to measure and represent dependencies, ranging from traditional correlation metrics to more advanced structures suited for capturing complex and tail-dependent behaviours.

3.1 Statistical Meaning and Importance

In the context of risk modelling, particularly in insurance and banking, understanding and accurately representing dependencies among risks is of paramount importance. Risk modelling typically involves two foundational components:

- The marginal distribution of each individual risk;
- and the dependency structure that links these marginal distributions.

While estimating marginal risk distributions is already a complex task, modelling dependencies presents an even greater challenge. This is because dependencies encapsulate the systemic, often hidden, relationships between risks, that can significantly alter the overall risk profile of an enterprise [19].
Dependencies can emerge due to macroeconomic factors such as inflation, interest rates, and exchange rates, which affect both assets and liabilities. For instance, inflation can erode the real value of assets while increasing the cost of claims, especially in long-tail insurance lines, thereby

impacting both sides of the balance sheet. Likewise, dependencies may emerge from common exposure scenarios across different lines of business; for example, a single catastrophic event like a hurricane can simultaneously impact P&C (property and casualty) and life insurance portfolios.
The danger of oversimplifying these interdependencies is that a model may produce an overly optimistic picture of a firm's overall risk, even when each marginal component appears reasonably assessed. This issue becomes especially relevant when dependencies intensify during stressed market conditions, a phenomenon well-documented during financial crises, where risk factors that once appeared uncorrelated suddenly moved in the same direction. Therefore, capturing and accurately representing dependencies is crucial for a realistic evaluation of enterprise-wide risk.

3.2 Mathematical Representation of Dependency

Ideally, one would represent all dependencies via a comprehensive system of equations, capturing every causal and correlative link. However, such a system is not only infeasible to construct but also impossible to parametrise accurately with available data. Consequently, statistical modelling relies on more tractable, but often imperfect, tools to approximate these relationships.
Traditionally, dependency between two risks is quantified using the linear correlation coefficient, a single scalar statistic. While useful in certain contexts, linear correlation is inadequate for capturing the full range of dependency structures. The term dependency structure is preferred over mere correlation when relationships are non-linear or when dependencies vary across the distribution, particularly in the tails. For instance, two risks might show moderate average correlation but exhibit near-perfect dependency in extreme loss scenarios. This tail dependency is especially relevant in economic capital modelling, where extreme outcomes drive capital requirements.

3.3 Types of Dependency Measures

To better understand the nature of dependence between financial variables, it is essential to distinguish among different types of dependency measures. These measures can be broadly categorised into two families: those based on linear correlation and those that rely on rank-based correlations, which are capable of capturing monotonic but potentially non-linear relationships. Figure 3 illustrates this classification, highlighting Pearson correlation for linear associations, and Spearman's rho and Kendall's tau for rank-based associations. Each of these measures offers distinct insights into dependency structure, and the choice of measure should be guided by the nature of the underlying data and the objectives of the analysis. In the following subsections, as described by Shaw and Spivak [48], we discuss the main ones.

3.3.1 Linear Correlation Coefficient

The Pearson linear correlation coefficient remains the most widely recognised measure of dependency. Defined for pairs of random variables X,Y with finite variance, it quantifies the degree of linear association:

$$\rho^{Pearson} = Cor\left(X,Y\right) = \frac{\mathbb{E}\left[\left[X-\mathbb{E}\left(X\right)\right]\left[Y-\mathbb{E}\left(Y\right)\right]\right]}{\sqrt{Var\left(X\right)Var\left(Y\right)}}. \tag{1}$$

A value of +1 or -1 indicates a perfect increasing or decreasing linear relationship, respectively, while a value of 0 indicates no linear relationship.
However, linear correlation has several critical limitations:

- It detects only linear relationships and fails to capture non-linear dependencies;
- A zero correlation does not imply independence;
- It is sensitive to the marginal distributions of the variables;
- It is not invariant under non-linear transformations;
- It requires finite variance, making it unsuitable for heavy-tailed distributions, like Lévy or Pareto, common in financial modelling.

These deficiencies limit the usefulness of correlation in capturing complex, non-linear and tail-dependent behaviours often seen in real-world risk data.

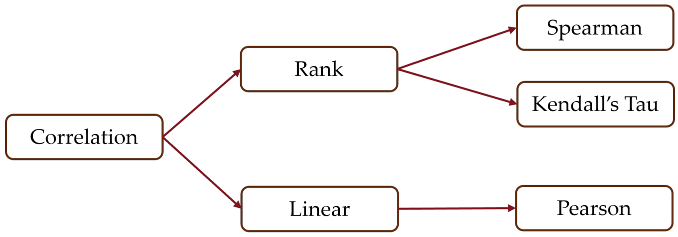


FIGURE 3: Different types of correlations

3.3.2 Rank Correlation Measures

To address these limitations, rank-based correlation measures such as Spearman's rho and Kendall's tau have been developed. These statistics measure the strength of a monotonic relationship between variables based on the ranks rather than the raw values.

Spearman's rho
Spearman's rank correlation coefficient is the linear correlation between the ranked values of two variables:

$$\rho^{Spearman} = \rho\left(F_X\left(X\right), F_Y\left(Y\right)\right), \tag{2}$$

where F_X and F_Y are the cumulative distribution function (i.e. ranks) of X and Y.
Unlike Pearson's correlation, which captures only linear relationships, Spearman's coefficient measures the strength of monotonic associations by assessing how consistently one variable increases or decreases with the other. It is invariant under monotonic transformations and it does not need any assumption about the distribution, making it a flexible choice when data do not meet the hypothesis of linear correlation [50].

Kendall's tau
Kendall's tau, is another non-parametric measure that assesses the concordance between pairs of observations. Given a sample of n paired observations, define:

- C : number of concordant pairs;
- D : number of discordant pairs;
- $S = C - D$.

Then:

$$\tau = \frac{2S}{n\left(n-1\right)}. \tag{3}$$

It provides a more intuitive understanding of dependency, quantifying the probability that variables move in the same direction [39].
Both measures share the advantage of being distribution-free, they do not rely on any assumptions about the marginal distributions of the variables. This property makes them particularly useful in non-parametric settings and for calibrating copulas from empirical data.
Nevertheless, these rank-based measures are not without their own shortcomings, which partly explains their less frequent use in industry practice. First, rank correlations are less interpretable in economic or financial terms: while a Pearson correlation can be readily understood as a linear link, a Spearman correlation refers to a monotonic trend but does not convey the magnitude of change. Second, these measures are less tractable analytically, especially in high-dimensional settings, making it harder to integrate them into standard statistical and actuarial models.
Moreover, rank-based measures are often less efficient when the true relationship between variables is linear. In such cases, Pearson correlation provides more statistical power. Computationally, calculating Kendall's tau can also be intensive for large datasets, as it involves pairwise comparisons, making it impractical for some large-scale applications.
In practice, while rank correlations are invaluable for understanding non-linear or ordinal relationships, their limited adoption stems from practical, interpretative, and computational challenges,

rather than from lack of merit.
As risk management evolves, it becomes increasingly critical to move beyond simple correlation metrics and adopt more sophisticated dependency structures that reflect the real-world behaviour of risks, especially under stress scenarios.

3.4 Beyond Correlation: Introduction to Copulas

Copulas provide a sophisticated mathematical framework to describe the dependency structure between random variables independently from their marginal distributions [44]. In risk modelling, this aligns well with the typical two-step process: first, modelling each individual risk's marginal distribution and, second, specifying how these risks interact or co-move, especially under stress. The second step is where copulas offer distinct advantages over correlation matrices.
Sklar's Theorem underpins the use of copulas by demonstrating that any multivariate joint distribution can be decomposed into its marginal distributions and a copula that binds them together [42]. This allows for the construction of joint distributions with specified marginal behaviours and tailored dependency characteristics, making copulas indispensable in the modelling of aggregate risk.

3.4.1 Copula Mathematics

A copula is a multivariate distribution function defined on the unit hypercube $[0,1]^n$, which captures the dependence structure between random variables, independent of their marginal distributions. Specifically, an n-dimensional copula is a joint distribution function $C(u_1, \dots, u_n)$ of a random vector $(U_1, \dots, U_n)$, where each component U_k is uniformly distributed on $[0,1]$. This means that for all $k = 1, \dots, n$, the marginal distribution satisfies:

$$P(U_k \leq u) = u \text{ for all } u \in [0,1].$$

Definition 3.1 (Copula). *A function $C : [0,1]^n \to [0,1]$ is called a* copula *if it satisfies the following properties:*

- $C(u_1, \dots, u_n) = 0$ *whenever at least one $u_i = 0$. This ensures the distribution function starts at zero;*
- *For every $i \in \{1, \dots, n\}$,*
$$C(1, \dots, 1, u_i, 1, \dots, 1) = u_i,$$
where all arguments are 1 except the i-th coordinate;
- *The function C is n-increasing.*
For example, in the bivariate case ($n = 2$), for all $(a_1, a_2), (b_1, b_2) \in [0,1]^2$ such that $a_k \leq b_k$ for $k = 1, 2$, we have
$$C(b_1, b_2) - C(a_1, b_2) - C(b_1, a_2) + C(a_1, a_2) \geq 0.$$
This condition ensures that C defines a valid joint cumulative distribution function.

3.4.2 Sklar's Theorem

Sklar's Theorem is a foundational result in the theory of copulas [49]. It provides a formal connection between multivariate distribution functions and their marginals.

Theorem 3.1 (Sklar's Theorem). *Let $F(x_1, \dots, x_n)$ be the joint cumulative distribution function of a random vector $(X_1, \dots, X_n)$, with marginal $F_1(x_1), \dots, F_n(x_n)$.*
Then, there exists a copula $C : [0,1]^n \to [0,1]$ such that for every $(x_1, \dots, x_n) \in \mathbb{R}^n$, the following holds:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

Moreover, if the marginal distributions $F_1, \dots, F_n$ are continuous, then the copula C is unique.

This theorem provides a mathematical framework that separates the modelling of the dependence structure (via the copula) from the modelling of the marginal distributions. That is, once the marginals are known or estimated, the copula aggregates them into a full joint distribution.
Furthermore, Sklar's Theorem implies an important invariance property: if C is a copula for the random vector $(X_1, \ldots, X_n)$, then for any set of strictly increasing transformations $T_1, \ldots, T_n$, the same copula C describes the transformed vector:

$$(T_1(X_1), \ldots, T_n(X_n)).$$

This invariance under strictly increasing transformations is analogous to properties of rank correlation measures like Kendall's tau and Spearman's rho, which are also invariant under monotonic transformations of the variables [17]. It makes copulas especially useful in modelling dependencies that are unaffected by scaling or marginal distributional changes.

4. Aggregation Methodologies

The aggregation of risks represents a crucial step in modern financial and actuarial modelling, bridging the gap between isolated marginal distributions and a consolidated assessment of total portfolio risk. This chapter explores the main methodologies used to aggregate risks across financial and insurance portfolios, with an emphasis on theoretical foundations, practical implementations and methodological rigour. A variety of methods have emerged, each with unique characteristics in terms of assumptions, computational complexity, regulatory acceptance and capacity to capture real-world dependencies. Figure 4 provides a high-level taxonomy of the principal methodologies employed in risk aggregation. These range from basic summation techniques to advanced structural models incorporating macroeconomic drivers. The classification illustrates how methodologies differ in complexity, modelling assumptions, and capacity to capture risk interdependencies.

4.1 Basic Aggregation Techniques and Limitations

Risk aggregation aims to provide a comprehensive understanding of overall portfolio exposure by combining multiple risk sources. The following discussion focuses on the insurance sector, particularly on the Solvency Capital Requirement (SCR), although similar reasoning can be applied to the banking sector.
The simplest approach is the summation of stand-alone capital requirements across risk types, which assumes perfect correlation (100%) and offers no diversification benefit. Formally, this is expressed as:

$$SCR_{agg} = \sum_i SCR_i. \tag{4}$$

Though conservative and easy to communicate, it significantly overstates capital needs and is seldom used in modern frameworks except for benchmarking. A marginal improvement is the use of a fixed diversification percentage, which adjusts the simple sum by a constant factor $k \in (0,1)$:

$$SCR_{agg} = k \cdot \sum_i SCR_i. \tag{5}$$

However, this method is largely static, ignores changing inter-risk dynamics and lacks statistical grounding [48]. Both techniques are considered rudimentary and are primarily found in legacy systems or for conservative buffers.
To overcome the limitations of such basic techniques, the variance-covariance matrix approach has become widely adopted. This methodology incorporates pairwise correlations between risk factors and aggregates capital using the well-known square-root formula:

$$SCR = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}, \tag{6}$$

where ρ_{ij} is the correlation coefficient between risks i and j, and SCR_i is the standalone capital for risk i. This formula assumes elliptical distributions and linear dependence. While efficient,

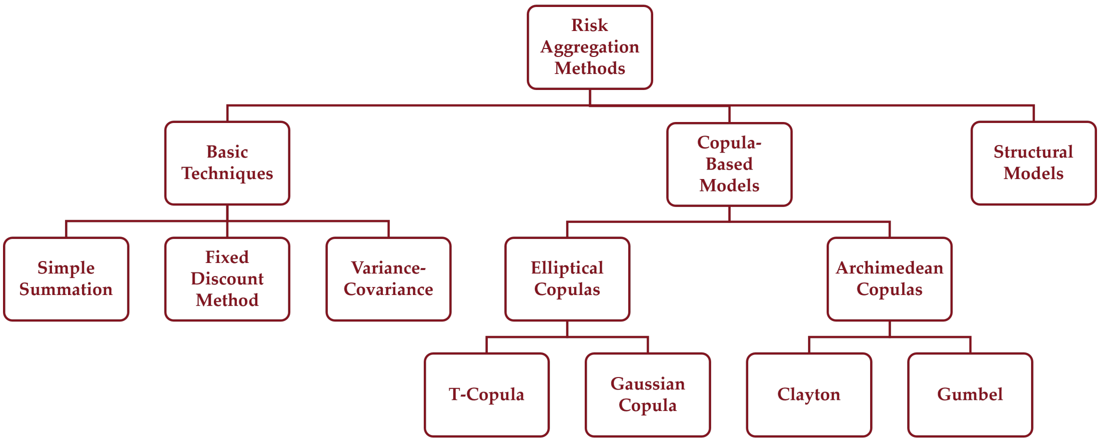


FIGURE 4: *Overview of Risk Aggregation Methods*

its effectiveness depends on the correlation matrix, that is symmetric and positive semi-definite. It is widely used in regulatory frameworks, notably in the Solvency II standard formula, which provides standardised correlation parameters across risk modules. Otherwise, in partial or full internal models, the ρ_{ij} can be estimated using statistical metrics as described in the previous chapter. However, the true dependence structure in extreme events may deviate from these assumptions, prompting concerns over model robustness.

4.2 Copula-Based Models and Scenario-Based Aggregation

Copula-based modelling offers a more flexible framework by decoupling the marginal behaviour of individual risks from the dependence structure that binds them.
The Gaussian copula is derived from the multivariate normal distribution. It is defined as:

$$C^{\text{Gauss}}(u_1, \ldots, u_d) = \Phi\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right), \tag{7}$$

where Φ^{-1} is the inverse of the standard normal cumulative distribution function, and Φ denotes the joint CDF of a multivariate normal distribution.
The Gaussian copula is popular due to its mathematical tractability and ease of simulation. However, a significant limitation lies in its lack of tail dependence. This implies that even extreme events in individual risks are not likely to occur simultaneously, an unrealistic assumption in stress scenarios or market crashes. As such, while the Gaussian copula works well under normal conditions, it tends to underestimate joint risk during crises.
The (Student's) t copula offers a remedy to the Gaussian copula's limitations by incorporating tail dependence. It is constructed analogously from the multivariate t-distribution and is defined as:

$$C^{t}(u_1, \ldots, u_d) = t_\nu\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\right), \tag{8}$$

where t_ν^{-1} is the quantile function of the univariate t-distribution with degrees of freedom ν and t_ν is the joint CDF of the multivariate t-distribution. This introduces a degree of freedom parameter ν, which governs the extent of tail dependence: lower values of ν produce stronger tail dependence, meaning that extreme co-movements become more likely. In copula-based modelling, as said in *Theorem 3.1*, the marginals and the copula are specified separately. The choice of marginal distributions is crucial, as it captures the individual behaviour of each risk. Common choices include the normal distribution for simplicity, or heavy-tailed distributions such as the Student's t or generalised Pareto when modeling financial or insurance data with extreme values. Importantly, the same copula can be combined with different marginals, allowing practitioners to tailor the model to the marginal properties observed in the data, independently of the dependence structure.

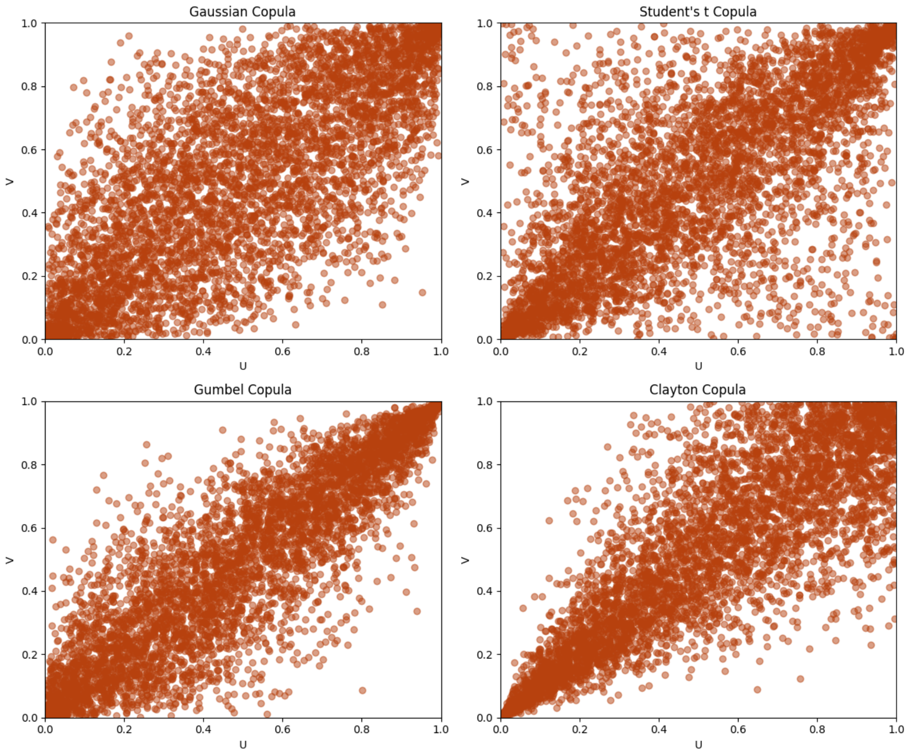


FIGURE 5: Different copulas

The t copula is elliptical and symmetric, implying that both upper and lower tails are treated equally. While this is an improvement over the Gaussian case, it can be limiting in applications focused on one tail only: in such cases, the symmetry of the t copula may lead to a misrepresentation of tail dependence, as its calibration is influenced equally by both tails. This can result in an inaccurate characterization of extreme co-movements in the tail of interest.
Thus, the t copula allows for a more realistic modelling of joint extremes, though it assumes uniform tail behaviour across all risk pairs.
While the Gaussian and t copulas are both elliptical and derived from multivariate distributions, Archimedean copulas offer a fundamentally different approach. They are typically simpler in form, highly flexible and particularly useful in modelling asymmetric dependency structures, including one-sided tail dependence.
Two prominent members of this family are the Clayton and Gumbel copulas. The Clayton copula is defined as:

$$C_\theta^{\text{Clayton}}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \theta > 0. \tag{9}$$

It exhibits lower tail dependence but no upper tail dependence. This makes the Clayton copula ideal for applications where simultaneous extreme losses (e.g., defaults) are of concern. In contrast, the Gumbel copula focuses on upper tail dependence, and is given by:

$$C_\theta^{\text{Gumbel}}(u,v) = \exp\left[-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right], \theta \geq 1. \tag{10}$$

This copula is particularly well-suited to modelling the co-occurrence of extreme gains or large claim events, as common in catastrophe insurance or systemic market rallies.
In selecting an appropriate copula for stress testing or risk modelling, no single choice universally outperforms the others. As shown in Figure 5, the Gaussian copula does not present clear evidence of tail dependence, the t copula, instead, captures symmetric tail dependence and Archimedean copulas like Clayton and Gumbel allow for asymmetric extremes. Although the Gaussian copula remains dominant due to its simplicity, tractability, and widespread adoption. A key advantage of elliptical copulas lies in their simulation simplicity, particularly in higher dimensions.

Algorithm 1 Simulation from an elliptical copula (Gaussian or t)

1: Choose an elliptical copula with parameters:
- correlation matrix Σ
- degrees of freedom ν (for t copula)

2: Specify the marginal distributions $F_1, \ldots, F_d$
3: Generate a random vector $\mathbf{z} = (z_1, \ldots, z_d) \sim N_d(0, \Sigma)$ or $t_{\nu,\Sigma}$
4: **for** each component $i = 1, \ldots, d$ **do**
5: Compute $u_i = \Phi(z_i)$ (or $u_i = t_\nu(z_i)$)
6: **end for**
7: The resulting vector $\mathbf{u} = (u_1, \ldots, u_d)$ follows the copula distribution
8: Transform using marginals: $x_i = F_i^{-1}(u_i)$

The sampling procedure is straightforward relying on transformations from multivariate distributions. This simplicity is especially attractive in stress testing frameworks, where large-scale scenario generation is often required. The pseudo-algorithm of a typical simulation process is outlined in Algorithm 1.
In contrast, simulation from Archimedean copulas, especially in more than two dimensions, is often more involved. While conditional methods or generator inversion techniques exist, they require sampling from non-standard distributions and are less efficient or less scalable in higher-dimensional settings.
This difference in computational tractability further supports the use of elliptical copulas in large-scale stress testing, even if they may be less expressive in capturing asymmetric tail behaviour.
Moreover, as highlighted by Koziol et al. (2015) [40], the Gaussian copula can generate severe stress scenarios when assuming extreme stress forecasts. This is because, although it lacks tail dependence in a global sense, it is an elliptical distribution, meaning that joint extreme events become more likely when analysis is focused on a narrow, stressed region of the tail. Consequently, with appropriate scenario design, the Gaussian copula remains a valid and effective choice in stress testing frameworks.

4.3 Structural Models and Scenario-Based Aggregation

Taking a more integrated approach, structural models simulate joint risk evolution by linking risk components through shared macroeconomic drivers. Consider an economic factor vector Z influencing each risk component via functions $X_i = f_i(Z)$.
To generate realistic economic scenarios for the factor Z, structural models rely on stochastic modelling frameworks. Common choices include Vector Autoregressive (VAR) models or Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models. More generally, scenario generation can also be based on stochastic differential equations frameworks (such as the Heston stochastic volatility or Merton structural credit models). The joint distribution of total risk is then obtained via Monte Carlo simulations or analytical approximations. Letting $SCR_i(Z)$ represent risk under scenario Z, the total risk becomes:

$$SCR = \mathbb{E}_Z \left[g \left(SCR_1(Z), \ldots, SCR_n(Z) \right) \right], \tag{11}$$

where g is an aggregation function. Structural models offer realism and scenario analysis capability but come with high data and model governance demands. Their application aligns with the ORSA (Own Risk and Solvency Assessment) principle and supports forward-looking solvency analysis.

4.4 Comparative Insights and Practical Considerations

Bringing these methodologies into perspective, it becomes clear that each presents distinct advantages and limitations. The summation and fixed discount approaches, while easy to apply and communicate, are overly conservative and ignore real diversification effects. In contrast, the variance-covariance method offers a balance of practicality and statistical grounding but remains sensitive

to the assumption of linearity. Copula-based models enable nuanced dependency modelling and provide greater flexibility in tail-risk analysis yet require careful calibration and pose model selection challenges. Structural aggregation models stand out for their depth and coherence, integrating macroeconomic linkages directly into risk projections. However, their demand for data, expertise, and computational power may pose barriers to adoption.
Overall, the selection of an aggregation methodology should align with the institution's risk profile, data environment and regulatory context. As regulatory expectations evolve, institutions increasingly adopt advanced frameworks validated through empirical testing and scenario analysis. A well-constructed aggregation strategy is essential for accurate capital assessment, product pricing and long-term financial resilience. Furthermore, collaboration between actuarial, risk management and regulatory functions enhances the transparency and reliability of aggregation outcomes, supporting both internal governance and external reporting obligations.

5. Tail Dependence Methodologies

Understanding and accurately capturing extreme co-movements between risk factors is a critical concern in modern financial and actuarial risk management. Standard correlation-based techniques, while useful in average-case scenarios, fail to describe the behaviour of risks in the extreme tails of distributions, precisely where systemic crises tend to manifest. This limitation can lead to the underestimation of joint losses, particularly in capital adequacy frameworks built on Value-at-Risk, Expected Shortfall, or Economic Capital (EC) [42].
To address this gap, the concept of tail dependence has emerged as a vital tool. Tail dependence quantifies the probability that multiple risks experience extreme outcomes simultaneously, offering deeper insights into contagion effects, systemic stress, and portfolio vulnerabilities.
In this chapter, we present a comprehensive treatment of tail dependence in the context of risk aggregation. We introduce both qualitative and quantitative measures of tail dependence, moving from intuitive visual diagnostics to formal asymptotic coefficients. We then conclude with advanced modelling frameworks, highlighting the role of copula models for simulating joint tail behaviour and Extreme Value Theory (EVT) for extrapolating beyond observed quantiles.

5.1 Qualitative Measures of Tail Dependence

Qualitative techniques provide intuitive and preliminary insights into the presence of tail dependence between risk factors. Although they do not yield formal numerical measures, these methods are essential for initial exploration of non-linear and extreme co-movement patterns that may not be visible through standard correlation analysis, providing the necessary intuition to motivate the use of formal quantitative tools [18].

5.1.1 Rolling Correlation

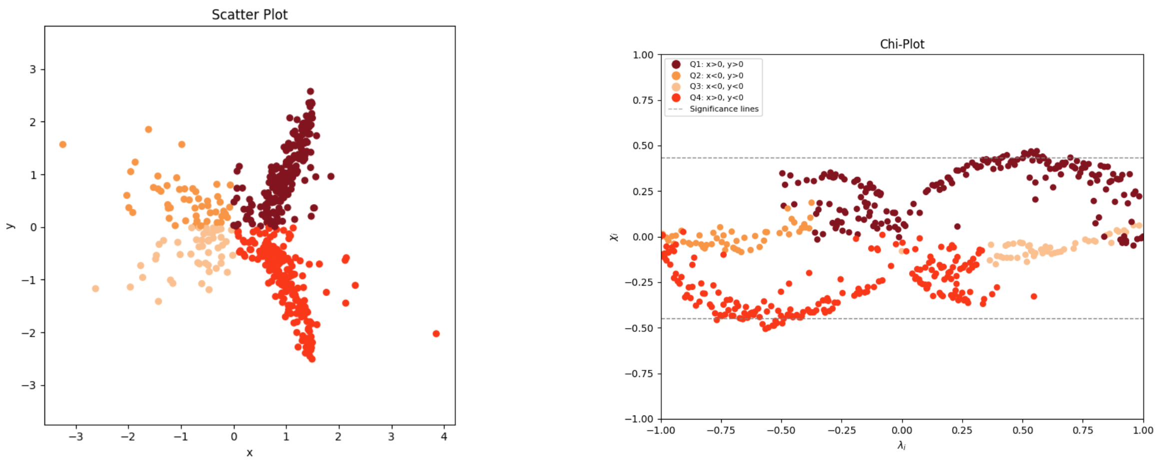
Rolling correlation is used to assess whether the correlation between two risk factors remains stable over time or varies during specific periods, such as market stress. For two time series X and Y, the rolling correlation over a window of size n at time t is defined by:

$$\rho_t^{(n)} = \frac{\sum_{i=0}^{n-1}(X_{t-i} - \bar{X}_t)(Y_{t-i} - \bar{Y}_t)}{\sqrt{\sum_{i=0}^{n-1}(X_{t-i} - \bar{X}_t)^2}\sqrt{\sum_{i=0}^{n-1}(Y_{t-i} - \bar{Y}_t)^2}}. \tag{12}$$

This moving estimate helps reveal temporal shifts in dependency structures, particularly relevant in financial systems where co-movements intensify under stress. However, rolling correlation primarily reflects average behaviour over short intervals and does not isolate dependence in the tails [42].

5.1.2 Scatter Plots

Scatter plots visually represent the joint distribution of two risk factors, as shown in Figure 6a. Under a Gaussian dependence structure, the observations tend to form elliptical contours, reflecting weak tail interaction. In contrast, tail-dependent distributions exhibit clustering in the joint corners,



(a) Scatter Plot from pseudo-random data

(b) Chi-Plot from pseudo-random data

FIGURE 6

such as the upper-right or lower-left quadrants [18]. This concentration suggests that extreme outcomes in both variables are likely to occur simultaneously.
While scatter plots are easy to generate and interpret, they do not offer a formal measure of dependence and are sensitive to sample size. Moreover, they are limited to bivariate analysis and may not distinguish between strong correlation and true tail dependence.

5.1.3 Chi-Plots

The Chi-plot provides a more advanced graphical tool to investigate non-linear dependence, as shown in *Figure 6b*. For a sample of observation pairs (x_i, y_i), the Chi-plot transforms ranks into coordinates (χ_i, λ_i) using empirical distribution functions [31]:

$$\begin{aligned} \chi_i &= \frac{H_i - F_i G_i}{\sqrt{F_i(1-F_i)G_i(1-G_i)}}, \\ \lambda_i &= 4S_i \cdot \max\left((F_i - 0.5)^2, (G_i - 0.5)^2\right). \end{aligned} \tag{13}$$

Here, H_i denotes the proportion of pairs (x, y) such that $x \leq x_i$ and $y \leq y_i$, and F_i, G_i are empirical cumulative probabilities of the data. S_i represents the sign of the quadrant (positive for the first and third, negative for the second and fourth). Large absolute values of χ_i indicate strong local dependence between the variables (in the example presented in Figure 6b, we considered values outside the 95% probability region to be significant, following the approach suggested in Fisher et al. (1985)[31]), while high values of λ_i emphasise observations that lie in the tails of the marginal distributions. This combination allows the Chi-plot to effectively detect tail-dependent structures.
Although Chi-plots reveal tail structure more clearly than scatter plots, they remain visual tools without a scalar summary statistic and are generally limited to bivariate cases.

5.2 Quantitative Measures of Tail Dependence

While qualitative techniques provide useful preliminary insight, they are limited by their subjectivity and lack of formal metrics. In contrast, quantitative measures offer a precise and replicable framework for assessing tail dependence are essential for building robust risk aggregation and capital allocation models. These measures aim to quantify the strength of joint extremes either through empirical probabilities or asymptotic coefficients derived from copulas.

Copula	Formula	λ_U	λ_L
Gaussian	$C^{\text{Gauss}}(u_1,\dots,u_d)=\Phi(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_d))$	0	0
Student-t	$C^t(u_1,\dots,u_d)=t_\nu(t_\nu^{-1}(u_1),\dots,t_\nu^{-1}(u_d))$	$2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$	$2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$
Gumbel	$C_\theta^{\text{Gumbel}}(u,v)=\exp\left[-\left((-\ln u)^\theta+(-\ln v)^\theta\right)^{1/\theta}\right],\quad \theta\geq 1$	$2-2^{\frac{1}{\theta}}$	0
Clayton	$C_\theta^{\text{Clayton}}(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta},\quad \theta>0$	0	$2^{-\frac{1}{\theta}}$

TABLE 2: Copulas and TDCs

5.2.1 Tail Dependence Coefficients

The most widely used theoretical measures of tail dependence are the tail dependence coefficients. They provide a formal way to quantify the asymptotic probability that one variable exceeds a high threshold given that the other does as well.
For a bivariate random vector (X, Y) with continuous marginal distribution functions F_X and F_Y, the upper and lower tail dependence coefficients are defined as:

$$\lambda_U = \lim_{u\to 1^-} \mathbb{P}(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)), \tag{14}$$

$$\lambda_L = \lim_{u\to 0^+} \mathbb{P}(Y < F_Y^{-1}(u) \mid X < F_X^{-1}(u)). \tag{15}$$

These limits capture the strength of co-movements in the tails of the distribution. If $\lambda_U > 0$, the X and Y are said to be asymptotically dependent in the upper tail; otherwise, they are asymptotically independent.
Tail dependence coefficients (TDCs) can also be expressed using the copula $C(u, v)$ of the joint distribution [49] [44]:

$$\lambda_U = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u}, \tag{16}$$

$$\lambda_L = \lim_{u\to 0^+} \frac{C(u,u)}{u}. \tag{17}$$

These coefficients are particularly useful for assessing dependence in risk aggregation contexts, as they isolate extreme co-occurrence from the overall dependence structure.

5.2.2 Parametric Estimators of Tail Dependence

Parametric estimators of tail dependence refer to analytical expressions for tail dependence coefficients derived under the assumption that the joint distribution of risk factors follows a known parametric family, such as the multivariate Student-t distribution or specific copula functions like Gumbel or Clayton.
These estimators do not constitute full dependence models themselves, but rather quantify tail dependence implied by specific parametric structures. While copulas play a central role in modelling, their tail properties can also be used through plug-in methods to estimate tail dependence coefficients based on fitted parameters.

Elliptical Distribution-Based Estimators
A foundational example of parametric estimation arises in elliptical distributions, particularly the bivariate Student-t distribution. Embrechts et al. (2002) [18] derive an expression for the upper tail dependence coefficient assuming that the pair (X, Y) follows a bivariate t-distribution with

correlation ρ and degrees of freedom ν. The resulting tail dependence coefficient is given by:

$$\lambda_U = 2t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right), \quad (18)$$

where $t_{v+1}(\cdot)$ is the cumulative function of the univariate Student-t distribution with $\nu+1$ degrees of freedom. This formula shows that tail dependence increases as either the correlation ρ increases or the degrees of freedom ν decrease. As $\nu \to \infty$, the Student-t distribution converges to the Gaussian distribution and the tail dependence coefficient tends to zero. This is consistent with the known property that Gaussian distributions imply asymptotic independence, even in the presence of high correlation.

Copula-Based Plug-in Estimators

Another widely used approach involves calculating tail dependence from the closed-form expressions available for known copula families. As Table 2 summarises, these estimators use the estimated copula parameters as inputs to derive the implied tail dependence. For example, the Gumbel copula, which is suited to modelling upper tail dependence, yields the coefficient:

$$\lambda_U = 2 - 2^{\frac{1}{\theta}}, \quad (19)$$

where $\theta \geq 1$ is the copula's dependence parameter. As θ increases, the upper tail dependence increases, approaching one in the limit. The Clayton copula, in contrast, is characterised by lower tail dependence. Its coefficient is given by:

$$\lambda_L = 2^{-\frac{1}{\theta}}, \quad (20)$$

where $\theta \geq 0$. In this case, lower values of θ correspond to weaker tail dependence. These expressions provide straightforward tools for quantifying the degree of dependence in specific parts of the joint distribution, which is particularly useful when the directionality of tail risk (e.g., lower vs upper) is known a priori, such as in credit risk or catastrophe insurance.

The Student-t copula, derived from the multivariate Student-t distribution, exhibits symmetric tail dependence in both the upper and lower tails. Its tail dependence coefficient is identical to that of the elliptical Student-t distribution discussed earlier. This makes the Student-t copula particularly attractive in financial risk modelling, where both large joint losses and gains are possible and need to be captured symmetrically.

In contrast, some copulas imply no tail dependence at all. The Gaussian copula, while widely used for its analytical tractability, always yields tail dependence coefficients of $\lambda_U = \lambda_L = 0$, regardless of the strength of correlation. This characteristic limits its applicability in scenarios involving systemic risk or stress testing, where co-extremes are likely [38] [44].

Plug-in Estimation Procedure

The parametric estimation process is typically carried out in two steps. First, the practitioner fits the chosen copula to data using methods such as maximum likelihood estimation or inversion of dependence measures like Kendall's tau. Second, the estimated parameter $\hat{\theta}$ is substituted into the copula's closed-form expression for λ_U or λ_L. This plug-in strategy provides computational efficiency and avoids the threshold selection problems inherent in non-parametric estimation.

However, it is important to recognise that parametric estimators are highly sensitive to model specification. Misidentifying the copula family or assuming elliptical dependence where none exists can lead to substantial under- or overestimation of tail risk. For this reason, parametric methods should be accompanied by goodness-of-fit tests, backtesting procedures, and, when necessary, supplemented with non-parametric or semi-parametric alternatives for robustness.

5.2.3 Non-Parametric Estimation of Tail Dependence

Non-parametric estimators of tail dependence provide a flexible, data-driven approach to measuring the strength of joint extremes without imposing strong assumptions about the underlying distribution or copula structure. These estimators are particularly useful when the true dependence

Copula	Analytical λ_U	Analytical λ_L	$\hat{\lambda}_U^{JOE}$	$\hat{\lambda}_U^{SS}$	$\hat{\lambda}_L^{SS}$	$\hat{\lambda}_L^{CG}$
Gaussian	0	0	0.065	0.065	0.053	0.053
Student-t	0.391	0.391	0.384	0.384	0.382	0.382
Gumbel	0.586	0	0.574	0.574	0.055	0.055
Clayton	0	0.707	0.003	0.003	0.716	0.716

TABLE 3: Non-Parametric Estimators Results

structure is unknown or complex, or when reliability is a priority, such as in regulatory contexts. Unlike parametric approaches, which rely on known functional forms and closed-form expressions, non-parametric methods infer tail dependence directly from empirical data using ranks, thresholds, and empirical copulas [37].

Empirical Copula-Based Estimator

One of the earliest and most intuitive non-parametric estimators of tail dependence is based on the empirical copula proposed by Joe et al. (1992) [37]. This estimator counts the frequency of joint exceedances in the upper tail and adjusts for the finite sample size. The upper tail dependence coefficient is approximated as:

$$\hat{\lambda}_U^{JOE} = 2 - \frac{1 - \hat{C}_n\left(1 - \frac{k}{n}, 1 - \frac{k}{n}\right)}{1 - \frac{k}{n}}, \tag{21}$$

where $\hat{C}_n(u, v)$ denotes the empirical copula evaluated at empirical quantiles, n is the sample size, and k is the number of exceedances considered (typically with $k \ll n$).

$$\hat{C}_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\#\{(x, y) | x \leq x_{(i)} \text{ and } y \leq y_{(j)}\}}{n}. \tag{22}$$

This estimator provides a simple way to assess whether extreme co-movements occur more frequently than under independence. While conceptually straightforward and widely applicable, the choice of k significantly affects the stability of the estimate, with smaller values improving extremality at the expense of higher sampling variability.

Schmidt-Stadtmüller Estimator

To address the trade-off between parametric precision and non-parametric robustness, Schmidt and Stadtmüller (2006) [47] proposed a semi-parametric estimator that combines rank-based empirical counting with asymptotic principles. The estimators are defined as:

$$\hat{\lambda}_U^{SS} = \frac{1}{k} \sum_{i=1}^{n} \mathbb{I}\left(R_i^{(1)} > n - k,\ R_i^{(2)} > n - k\right), \tag{23}$$

$$\hat{\lambda}_L^{SS} = \frac{1}{k} \sum_{i=1}^{n} \mathbb{I}\left(R_i^{(1)} \leq k,\ R_i^{(2)} \leq k\right), \tag{24}$$

where $R_i^{(1)}$ and $R_i^{(2)}$ are the marginal ranks of the i-th observation, and $k \in \{1, \ldots, n\}$. The indicator function $\mathbb{I}(\cdot)$ counts how many pairs of observations fall into the joint upper tail. This estimator has gained popularity in financial and insurance applications due to its conservative nature and ease of computation. It is especially well-suited to regulatory settings under Solvency II or ICAAP, where tail clustering must be captured without strong structural assumptions. It also forms the basis for several backtesting and model validation procedures in operational and market risk.

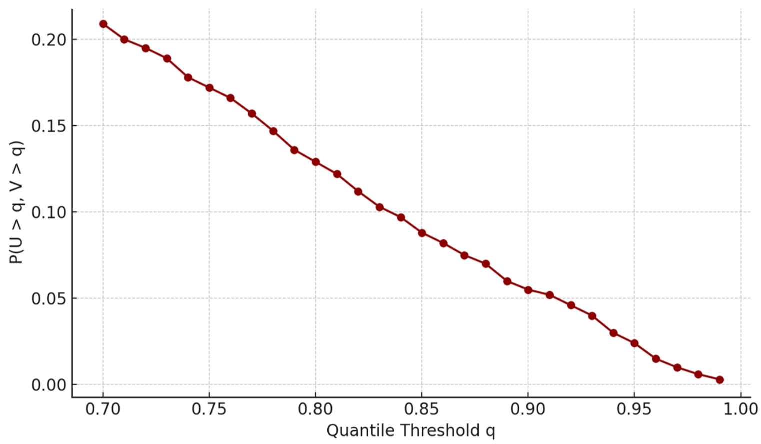


FIGURE 7: Empirical Joint Quantile Exceedance Probability (JQEP) on pseudo-random data

Caillault-Guegan Estimator
Beyond classical threshold-based estimators, other non-parametric techniques have emerged using the concept of empirical copulas. A notable contribution in this direction is the Caillault-Guegan estimator [11] , which defines a tail dependence estimator using:

$$\hat{\lambda}^{CG}_{L,n}\left(\frac{i}{n}\right) = \frac{\hat{C}\left(\frac{i}{n}, \frac{i}{n}\right)}{\left(\frac{i}{n}\right)}, \tag{25}$$

where $\hat{C}_n$ is the empirical copula. A 'plateau' detection algorithm identifies a stable zone over which estimates can be averaged:

$$\hat{\lambda}^{CG}_{L,n} = \frac{1}{p^*} \sum_{k=1}^{p*} \hat{\lambda}_{L,n}\left(\frac{k}{n}\right). \tag{26}$$

This estimator does not require assumptions about the marginal distributions and works well under weak structural assumptions. However, its reliability depends on carefully determining the smoothing range and identifying the stable zone correctly.

Implementation of Non-Parametric Estimators
All non-parametric estimators face common challenges, notably the sensitivity to the threshold k, the scarcity of tail observations, and the high variance of estimates in small samples. Careful selection of k, combined with bootstrap procedures or smoothing techniques, is often required to ensure stability and statistical reliability. Moreover, in high-dimensional contexts, the "curse of dimensionality" can affect the efficacy of non-parametric tail estimation, necessitating the use of pairwise or vine decomposition techniques to manage complexity.
To demonstrate the performance of the estimators in a controlled setting, we implemented an illustrative simulation study based on synthetic data generated from copula models. This framework allowed us to validate the non-parametric estimates against known analytical benchmarks. The threshold parameter k was selected as a simple yet representative choice that aligns with the theoretical considerations discussed by Schmidt and Stadtmüller (2006) [47], specifically $k = \lfloor \sqrt{n} \rfloor$. The corresponding results are presented in Table 3.
Despite these limitations, non-parametric estimators remain a cornerstone of practical tail risk assessment. Their model-free nature and interpretability make them valuable for initial diagnostics, model validation, and scenarios where robustness is prioritised over analytical convenience.

5.2.4 Joint Exceedance-Based Measures of Tail Dependence based

Joint exceedance-based measures assess tail dependence by directly examining the frequency of extreme co-movements between risk factors. These methods offer intuitive, empirical tools for analysing co-exceedances, which makes them especially useful in practical contexts such as systemic risk analysis, stress testing, and portfolio risk aggregation. By focusing on the likelihood that both variables exceed certain thresholds, joint exceedance measures can reveal hidden dependence in the tail that standard correlation methods fail to capture.

Joint Exceedance Probability (JEP)
The Joint Exceedance Probability (JEP) captures the likelihood that two variables exceed or fall below a given threshold simultaneously. For transformed uniform variables $U = F_X(X)$ and $V = F_Y(Y)$, the upper and lower tail JEPs are expressed as:

$$\begin{aligned} RJEP(z) &= \mathbb{P}(U > z,\ V > z), \\ LJEP(z) &= \mathbb{P}(U < z,\ V < z). \end{aligned} \tag{27}$$

In the case of independence, these probabilities reduce to $(1 - z)^2$ and z^2, respectively. Any upward deviation from these values is a sign of tail dependence [18]. A more risk-focused variant uses the quantile values of X and Y directly, as in:

$$JEP(\alpha) = \mathbb{P}(X > u_\alpha,\ Y > v_\alpha), \tag{28}$$

where $u_\alpha = VaR_X(\alpha)$ and $v_\alpha = VaR_Y(\alpha)$.

Tail Concentration Function
To refine the view further, Joe (1997) [38] introduced the Tail Concentration Function, which considers the conditional probability of joint exceedance:

$$R(z) = \frac{\mathbb{P}(U > z,\ V > z)}{1 - z},\ L(z) = \frac{\mathbb{P}(U < z,\ V < z)}{z}. \tag{29}$$

These are interpreted as $P(U > z|V > z)$ and $P(U < z|V < z)$, respectively. In the presence of tail dependence, these ratios will exceed the values implied by independence. In fact, the upper tail dependence coefficient can be retrieved as the limit $\lambda_U = \lim_{z\to 1} R(z)$, reinforcing the connection between this conditional probability and asymptotic dependence structure.

Joint Quantile Exceedance Probability (JQEP)
To address situations where different quantile levels must be used for each risk factor, the Joint Quantile Exceedance Probability (JQEP) generalises JEP. Its empirical form is:

$$JQEP_{empirical,\ lower}(p_x,\ p_y) = \mathbb{P}(F_X(X) < p_x,\ F_Y(Y) < p_y), \tag{30}$$

$$JQEP_{empirical,\ upper}\left(p'_x,\ p'_y\right) = \mathbb{P}\left(F_X(X) > p'_x,\ F_Y(Y) > p'_y\right).$$

Theoretical counterparts, based on copula models, are:

$$JQEP_{theoretical,\ lower} = \int\int_{(0,0)}^{(p_x,p_y)} C(u,v;\rho)\,du\,dv, \tag{31}$$

$$JQEP_{theoretical,upper} = \int\int_{(p'_x,p'_y)}^{(1,1)} C(u,v;\rho)\,du\,dv.$$

Here, the comparison between empirical and theoretical JQEP values helps assess the adequacy of the assumed dependence structure, particularly under a Gaussian or Student-t copula [42]. To further support the empirical interpretation of JQEP behaviour, Figure 7 displays a simulated chart of the empirical upper JQEP values plotted against increasing quantile thresholds. As expected, JQEP values drop sharply as the quantile threshold increases, consistent with the theoretical tail behaviour predicted by the copula model.

Conditional Quantile Exceedance Probability (CQEP)
A conditional version of this idea leads to the Conditional Quantile Exceedance Probability (CQEP),

which expresses the probability that one variable exceeds its quantile given that the other does. For the upper tail:

$$CQEP_{upper}(q) = \frac{\mathbb{P}\left(F_X(X) > q,\ F_Y(Y) > q\right)}{\mathbb{P}\left(F_Y(Y) > q\right)}. \tag{32}$$

Each of the quantitative methods presented provides a distinct lens on tail dependence. JEP and JQEP offer intuitive metrics based on observed co-exceedances. The Tail Concentration Function formalises these into conditional terms, while CQEP introduces directional insights. Finally, tail dependence coefficients offer asymptotic precision and theoretical elegance. These tools lay the foundation for estimation procedures and dependence modelling, particularly in the context of copulas.

Strengths and Limitations
Joint exceedance-based measures are highly intuitive and flexible, providing empirical tools to assess extreme co-movements in risk factors. These measures are particularly suited for backtesting and risk diagnostics, especially in cases where directional or asymmetric tail dependence is of interest. However, they do have limitations: their estimates can be noisy due to the scarcity of data in the tails, and they may not converge to the true tail dependence coefficients in small samples. To mitigate these issues, smoothing techniques and bootstrap methods are often employed, and high-dimensional dependence requires additional techniques like pair-copula constructions or vine copulas.
Despite these limitations, joint exceedance measures provide a valuable and practical set of tools for identifying tail dependence, especially when no explicit parametric model is available or desired. They complement more formal asymptotic measures and serve as a critical diagnostic in risk aggregation and capital modelling.

5.2.5 Copula Models as Generative Tools for Joint Tail Simulation

While parametric estimators based on copula functions offer closed-form expressions for tail dependence coefficients, copulas play a more fundamental role as generative models for simulating joint distributions. This modelling capability arises from Sklar's Theorem [49], as presented in chapter "Sklar's Theorem", which separates the marginal distributions from the dependence structure, allowing for flexible construction of multivariate models.
This approach enables the simulation of risk vectors that conform to both the empirical marginals and a chosen dependence structure, making it particularly valuable for modelling joint loss scenarios in risk aggregation, stress testing, and solvency assessment.
For high-dimensional settings, the copula modelling approach can be extended using vine copulas, which allow for the construction of multivariate copulas from a cascade of bivariate building blocks. This method enables scalable yet flexible modelling of complex dependence structures, accommodating tail asymmetries and conditional relationships [1]. Vine copulas are especially relevant in insurance portfolios, operational risk modelling, and multi-line product risk management, where pairwise tail behaviour plays a critical role.
Model calibration typically involves estimating the marginals independently, potentially using Extreme Value Theory (EVT) or generalised Pareto distributions, and then fitting the copula parameters using maximum likelihood methods, inference functions for margins (IFM), or inversion of rank-based measures such as Kendall's tau. Once calibrated, the model can be used to simulate multivariate loss distributions, compute joint Value-at-Risk, Expected Shortfall, or assess diversification benefits under regulatory frameworks such as Solvency II and Basel III [42][18].
Unlike tail dependence coefficients, which provide summary measures of joint extremes, copula models offer a full probabilistic representation of the dependence structure. They allow practitioners not only to quantify, but also to generate, realistic tail-dependent scenarios, a capability essential for solid capital estimation and systemic risk analysis.

5.2.6 EVT-Based Models and Extreme Quantile Analysis (EQA)

While copula-based models offer a versatile generative framework for simulating dependent risks, they remain limited by their reliance on a specific dependence structure and finite-sample cali-

brations. In contrast, EVT provides a complementary, asymptotic framework for modelling the behaviour of tail events without requiring a predefined copula family. EVT-based methods focus directly on the statistical behaviour of rare and extreme outcomes, making them particularly valuable for risk aggregation in sparse or highly volatile regimes.
EVT approaches model the tails of distributions either through the block maxima method, which relies on the Generalised Extreme Value (GEV) distribution, or the more flexible Peaks Over Threshold (POT) method, which models exceedances above a high threshold using the Generalised Pareto Distribution (GPD).
In the POT framework, the conditional distribution of excess losses $X - u$, given $X > u$, is approximated as:

$$\mathbb{P}(X > x \mid X > u) \approx \left(1 + \xi \frac{x - u}{\beta}\right)^{-1/\xi}, \quad \text{for } x > u, \tag{33}$$

where ξ is the shape parameter and β the scale parameter. This provides a flexible tool for modelling heavy-tailed behaviour in a univariate or marginal setting.
To extend this framework to multivariate and joint risk analysis, EVT can be integrated with copula techniques or applied through multivariate EVT constructions, such as multivariate threshold exceedances or angular measure models [32]. These allow the capture of extremal dependence beyond the limitations of traditional tail dependence coefficients. For example, even in cases where the upper tail dependence coefficient $\lambda_U = 0$, EQA may still uncover joint explosive behaviour in the far tails. This is particularly relevant for systemic risk modelling or operational loss estimation in banking and insurance [14] [42].
As a risk aggregation perspective, EVT-based methods may still detect joint clustering of extremes, a phenomenon especially relevant in systemic risk modelling, market crashes, or operational event losses [14].
EQA is a particularly powerful application of EVT in practice. EQA is designed to quantify loss behaviour beyond extreme quantile thresholds, typically beyond the 99.5th or 99.9th percentile, and is increasingly used in both Basel and Solvency II contexts to support conservative capital estimation. It relies on extrapolation methods that go beyond observed data, estimating the tail distribution in regions of data sparsity, where traditional empirical measures are insufficient.
In the multivariate context, EQA employs techniques such as conditional multivariate tail modelling, extremal coefficients, or hybrid EVT-copula approaches to better understand the shape and concentration of tail risk. These models enable more realistic joint loss distribution estimation and are particularly effective in capturing non-linear dependence structures during stress conditions.
However, the application of EVT and EQA is not without challenges. Threshold selection is a critical issue, as poorly chosen thresholds can lead to instability and high variance in parameter estimates. Diagnostic tools-such as mean residual life plots and stability plots-are commonly used to assess the suitability of threshold choices. Moreover, EVT estimators tend to be sensitive to sample size, and estimation uncertainty must be addressed, especially in risk contexts with limited tail observations [15] [32].
In summary, Extreme Quantile Analysis strengthens the risk quantification toolkit by allowing practitioners to capture asymptotic tail behaviour in both univariate and multivariate settings. Its integration with copula modelling or as a standalone approach provides a critical lens for understanding systemic and co-extreme risks, especially in regimes where traditional dependence measures fail to capture explosive joint behaviour.

5.3 Applications of Tail Dependence in Practice

The relevance of tail dependence modelling extends beyond theoretical interest; it plays a central role in practical risk management and regulatory decision-making. This subsection highlights how tail dependence concepts are effectively applied in the insurance, banking, and pension sectors, as well as in supervisory frameworks.

5.3.1 Insurance and Reinsurance

In the insurance domain, understanding joint tail behaviour is critical when aggregating diverse risk types such as mortality, longevity, and catastrophe risk. The European Solvency II directive [26] emphasises the importance of modelling tail dependencies, particularly when aggregating risks under the Solvency Capital Requirement (SCR) . Ignoring these dependencies can result in substantial underestimation of capital requirements, especially when risks are heavy-tailed and not linearly correlated. Copula-based approaches and extreme quantile techniques have been increasingly used to assess concentration effects and ensure sufficient capital buffers.

5.3.2 Banking Sector

In banking, tail dependence plays a pivotal role in stress testing and capital planning. Traditional correlation-based models often underestimate the likelihood of joint extreme losses across asset classes or sectors. Tail-dependent models, such as those involving the Student-t copula or empirical tail copulas, have been applied to assess contagion risk and systemic vulnerabilities. For example, during the global financial crisis, significant tail co-movements among credit instruments (e.g., reflected in iTraxx Crossover spreads) highlighted the necessity of using tail-aware models. Such methodologies inform the calculation of metrics like Value-at-Risk, Expected Shortfall, and CoVaR in portfolio- and institution-level risk aggregation [42] [38].

5.3.3 Pension Funds and Long-Term Investment Strategies

In pension fund management, understanding the tail behaviour between liabilities (e.g., longevity risk) and assets (e.g., equity risk) is crucial for long-term solvency planning. Tail dependence modelling supports the calibration of resilience scenarios, especially under shocks such as simultaneous market downturns and increases in life expectancy, which are better captured through dependence structures that extend beyond linear correlation [45].

5.3.4 Regulatory Guidelines and Supervisory Practice

Supervisory bodies, including the Bank for International Settlements (BIS), EIOPA, and the International Association of Insurance Supervisors (IAIS), explicitly caution against reliance on linear correlation for aggregating risks under stress [25] [33]. They recommend the use of models that incorporate tail dependence, particularly in the context of Pillar II evaluations, group supervision, and macroprudential surveillance. In operational risk and internal model validation, regulators increasingly request evidence that risk aggregation captures joint tail behaviour using credible, empirically supported methods.

The integration of tail dependence in real-world risk aggregation frameworks enables more resilient capital planning, accurate solvency estimation, and better preparation for systemic shocks. Whether through copulas, extreme quantile methods, or hybrid EVT frameworks, these tools help financial institutions move beyond naive assumptions of independence or normality, aligning modelling practices with the complex realities of joint risk behaviour.

6. Impact Analysis of Tail Dependence

As introduced in chapter "Risk Aggregation: Regulatory Evolution", capital requirements are a foundational pillar of financial regulation, designed to ensure the solvency and resilience of institutions in the face of adverse events. Both banking and insurance sectors have long operated under risk-based capital frameworks, which determine how much capital an institution must hold against its aggregate risks. Central to this process is the modelling of dependencies between different risk types. Traditionally, regulatory and internal models have relied on correlation-based methods for aggregating risk exposures. However, as recent literature and empirical studies demonstrate, the choice of dependence structure has a profound impact on capital calculations, risk governance, and ultimately on the stability of financial institutions.

This chapter explores the critical role that dependency structures play in shaping diversification benefits and capital requirements, with a particular focus on tail dependence. We draw on quantitative tools, empirical findings, and conceptual frameworks to examine the material differences that emerge when correlation-based models are replaced with more advanced, tail-sensitive alternatives such as copula models.

6.1 The Limits of Correlation-Based Dependence Structures

Under both Solvency II and Basel III, risk aggregation is typically performed using linear correlation matrices. These approaches are popular due to their mathematical simplicity and computational efficiency. However, the use of Pearson correlations inherently assumes linearity and elliptical distributions, which fails to capture the behaviour of extreme co-movements between risks, particularly during financial crises.
The CEIOPS in its *Calibration of the Solvency II standard formula: Technical Paper* [16] revealed that using a correlation matrix led to a market risk SCR of 61.9 (normalised units), whereas incorporating tail dependence raised the requirement to 82.5, showing a 33% underestimation when tail dependence was ignored.
This underestimation becomes particularly dangerous in times of systemic crises, when multiple risk factors become highly interdependent. Furthermore, correlation structures often produce misleading assessments of diversification benefits. During benign periods, linear correlations suggest substantial diversification, which may not materialise under stress. This mismatch between modelled and actual behaviour has prompted increasing scrutiny from regulators and internal model validation teams.

6.2 The Role of Tail Dependence and Copula Models

To address these limitations, modern risk aggregation has shifted toward the use of copula models, as presented in chapter "Measures of Dependency", which allow for more flexible and realistic dependence structures. Copulas separate the marginal behaviour of individual risks from their joint dependence, enabling a tailored representation of tail events.
For instance, Tang and Valdez (2005) [51] examined Australian general insurance data and found that capital requirements computed at the 99.5% confidence level varied from 92% to 101% of premiums depending on the copula used. Heavy-tailed copulas like the Student-*t*, which model tail dependence, led to the highest capital charges. The authors conclude that improper modelling of tail dependence can materially distort capital and diversification outcomes.
In the insurance context, Mejdoub and Ben Arab (2018) [43] used a D-vine copula model on non-life portfolios and showed that ignoring tail dependence in aggregation can significantly distort Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR), leading to under- or overestimation of required capital depending on the dependence structure.
These findings are not merely academic. They have concrete implications for solvency, capital planning, and regulatory compliance. Institutions using copula-based models benefit from a more realistic depiction of joint loss events, enhancing their resilience to systemic shocks.

6.3 Dependency Structures and Diversification Efficiency

One of the most critical aspects of capital modelling is the assessment of diversification benefits. While diversification is generally seen as a source of capital efficiency, its measurement is highly sensitive to how dependencies are modelled.
Correlation-based models often overstate diversification effects, particularly in portfolios exposed to multiple interacting risks. This can lead to underestimation of capital needs during market turbulence, where dependencies between risk factors intensify.
Rosenberg and Schuermann (2006) [46] analyzed integrated credit and market risks using copulas and showed that neglecting dependency across asset classes can lead to capital overestimation by 30–40% when no diversification is assumed. Though the Student-t copula framework improved tail sensitivity, the empirical increase in capital was modest in their baseline application, emphasizing the role of dependence assumptions on diversification measurement.
Copula models, by contrast, can differentiate between normal and stressed conditions. They reveal

that under tail dependence, diversification deteriorates as risks begin to move together, particularly in the upper quantiles of the loss distribution. This is where capital adequacy is truly tested.

6.4 Analytical Framework for Dependency Impact

To operationalise the evaluation of dependency structures, internal models have developed diagnostic tools that provide deep insight into the sensitivity and concentration of capital requirements. These tools are particularly useful for understanding how different risk components and assumptions about their interrelations affect overall capital needs. While this approach can be applied across financial sectors, this section focuses specifically on the insurance context, following the approach developed by EIOPA in its *Study on Diversification in Internal Models* [25].

Risk Multiplier
The risk multiplier quantifies how a single undiversified risk component impacts total capital under a variance-covariance method:

$$\text{riskMultiplier}_j = \sum_i \rho_{ij} \cdot \frac{\text{undivSCR}_i}{\text{SCR}^{\text{vc}}}. \tag{34}$$

A high multiplier implies both a significant standalone risk and strong interactivity with other components. For example, if market risk shows a 50% multiplier, even moderate shifts in its value can materially affect the total SCR, guiding portfolio rebalancing and additional buffer provisioning.

Correlation Multiplier
This metric measures how sensitive the SCR is to changes in specific correlations:

$$\text{correlationMultiplier}_{ij} = \frac{\text{undivSCR}_i}{\text{SCR}^{\text{vc}}} \cdot \frac{\text{undivSCR}_j}{\text{SCR}^{\text{vc}}}. \tag{35}$$

This metric is crucial for assessing model risk. Internal models often rely on subjective or expert-based assumptions for correlation matrices, especially where historical data is sparse or unreliable. A high correlation multiplier signals that even a small misspecification in correlation can have a disproportionately large impact on capital. In practice, this helps model owners and validators to identify where careful justification and stress testing is needed.
For example, if the correlation between market and non-life risks has a correlation multiplier of 25%, then an optimistic assumption could understate capital significantly. This metric thus anchors the model to reality-checks and encourages prudent assumptions.

Diversification Benefit Decomposition
Diversification is often presented as a total benefit, but this metric dissects it risk by risk, providing a nuanced view of how each contributes to the total capital relief.
Key metrics involved are:

- **Undiversified weight**: Share of the undiversified SCR each risk contributes.
- **Diversified weight**: Share of the diversified SCR.
- **Individual diversification benefit**: How much the capital for that risk is reduced due to diversification.
- **Diversification benefit weight**: Each risk's share of the total benefit.

$$\text{divBenefit}_i = 1 - \frac{\text{divSCR}_i}{\text{undivSCR}_i}, \tag{36}$$

$$\text{divBenefitWeight}_i = \frac{\text{undivSCR}_i - \text{divSCR}_i}{\sum_j (\text{undivSCR}_j - \text{divSCR}_j)}. \tag{37}$$

This disaggregation allows to highlight which risks generate most of the diversification benefit and which contribute least. A risk with a high undiversified weight but a low diversification benefit may be a concern, particularly if it dominates the capital structure. Conversely, a high diversification benefit may validate the strategic value of holding that risk in the portfolio.

6.5 Quantile-Based and Tail Analysis

In chapter "Tail Dependence Methodologies", we introduced Extreme Quantile Analysis (EQA) as a powerful extension of traditional tail risk measurement, grounded in Extreme Value Theory (EVT). Here, we apply it alongside Landing Quantile Analysis to deepen our understanding of diversification under extreme stress.

Extreme Quantile Analysis applies Extreme Value Theory (EVT) to assess whether internal models realistically capture the far right tail of the loss distribution, typically beyond the 99.5th percentile. This is essential for identifying underestimation of rare, high-impact events like pandemics or financial crises.

Landing Quantile Analysis evaluates where the diversified SCR lies within the empirical loss distribution of a given risk. A high landing quantile suggests poor diversification, possibly due to persistent tail dependence, whereas a lower quantile implies that diversification is effectively reducing capital needs.
When performed across all risk types, landing quantile analysis provides a map of diversification efficiency, helping identify risks that behave poorly under aggregation or remain problematic in stress scenarios.

6.6 Dependency Structures and Capital Composition

Another layer of analysis tracks how risk composition changes across quantiles of the loss distribution. At the median or lower quantiles, risk contributions may appear balanced, but at higher quantiles, such as the 99.9th percentile, certain risks may dominate. This shift highlights how apparent diversification can collapse under stress.
Smoothing techniques such as Gaussian kernels or Harrell-Davis estimators help visualise this dynamic distribution of risk, offering a more accurate picture of portfolio behaviour under adverse scenarios.

6.7 Measuring Concentration and Diversification Limits

Even the best dependency modelling cannot overcome structural concentration. The Gini coefficient offers a quantitative measure of how evenly risks contribute to the portfolio:

$$G = \frac{\sum_i \sum_j |x_i - x_j|}{2n \sum_i x_i}, \quad \text{where} \quad x_i = \frac{\text{undivSCR}_i}{\sum_j \text{undivSCR}_j}. \tag{38}$$

A coefficient near zero implies broad diversification, while a value closer to 1 indicates that the portfolio is dominated by one or two risks, reducing the potential for effective capital relief.
This metric provides a realistic ceiling to diversification: even with ideal dependency assumptions, a highly concentrated portfolio cannot achieve strong capital efficiency.
To complement this, the **Diversification Score** compares actual diversification to the theoretical maximum under full independence:

$$\text{divScore} = \frac{\sum_i \text{undivSCR}_i - \text{divSCR}_{\text{total}}}{\sum_i \text{undivSCR}_i - \sqrt{\sum_i \text{undivSCR}_i^2}}. \tag{39}$$

A high score suggests that the portfolio is well-diversified relative to its potential, given the existing risk profile. A low score, on the other hand, may signal that correlation assumptions are overly conservative or that the model is failing to recognise legitimate diversification effects.
For peer comparisons, this score helps benchmark institutions with similar diversified profiles but differing dependency structures. It provides a clean, normalised view of how effectively a firm is leveraging diversification.

6.8 Final Considerations

This chapter has shown that the structure of dependency modelling is not a technical footnote but a defining factor in the accuracy and prudence of capital requirement calculations. Traditional correlation-based models offer convenience but often fail to capture the true nature of joint tail events, particularly under stress. Copula-based approaches, though more complex, provide a much richer and more realistic framework for risk aggregation, resulting in higher, but more appropriate, capital requirements.

By applying quantitative diagnostics such as risk and correlation multipliers, quantile-based tail analyses, and concentration metrics, institutions can rigorously evaluate their internal models and improve their risk governance. Ultimately, the choice of dependency structure determines whether capital models merely comply with regulation or genuinely safeguard financial stability. As systemic risks evolve and regulatory scrutiny deepens, the ability to model and manage tail dependencies will be increasingly critical in the financial industry's toolkit.

7. Conclusions

The progressive evolution of regulatory frameworks within the banking and insurance sectors underscores a growing awareness of the pivotal role that tail dependence plays in the accurate quantification of systemic risk. While traditional correlation-based models remain foundational, they often fail to capture the complex interdependencies that emerge during extreme market events. As a result, regulators are increasingly advocating for more sophisticated approaches that consider the full spectrum of dependence structures, particularly in the tails.

This paper highlights the need to move beyond conventional correlation by examining alternative dependence modelling techniques, such as copula-based methods and other nonlinear association measures. These approaches offer a more realistic representation of how risks interact under stress. In particular, aggregation methodologies reveal how assumptions about dependence can materially influence capital adequacy assessments. In this light, the selection of appropriate tail dependence metrics and estimators is not a merely technical concern: it carries substantial regulatory and financial implications.

Looking forward, the future of prudential regulation will increasingly depend on the ability of both supervisory authorities and financial institutions to integrate methodological innovation into risk management practices. Furthermore, as financial systems grow more interconnected and exposed to tail risks, driven by global economic, climatic, and technological shifts,the ability to anticipate and mitigate the consequences of joint extreme events will become central to regulatory design.

A deeper understanding of tail dependence can strengthen the resilience of individual institutions and safeguard the broader stability of the financial system. Achieving this will require a thoughtful integration of advanced analytical methods with transparent and practical frameworks, capable of adapting to the complexities of a shifting risk environment.

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