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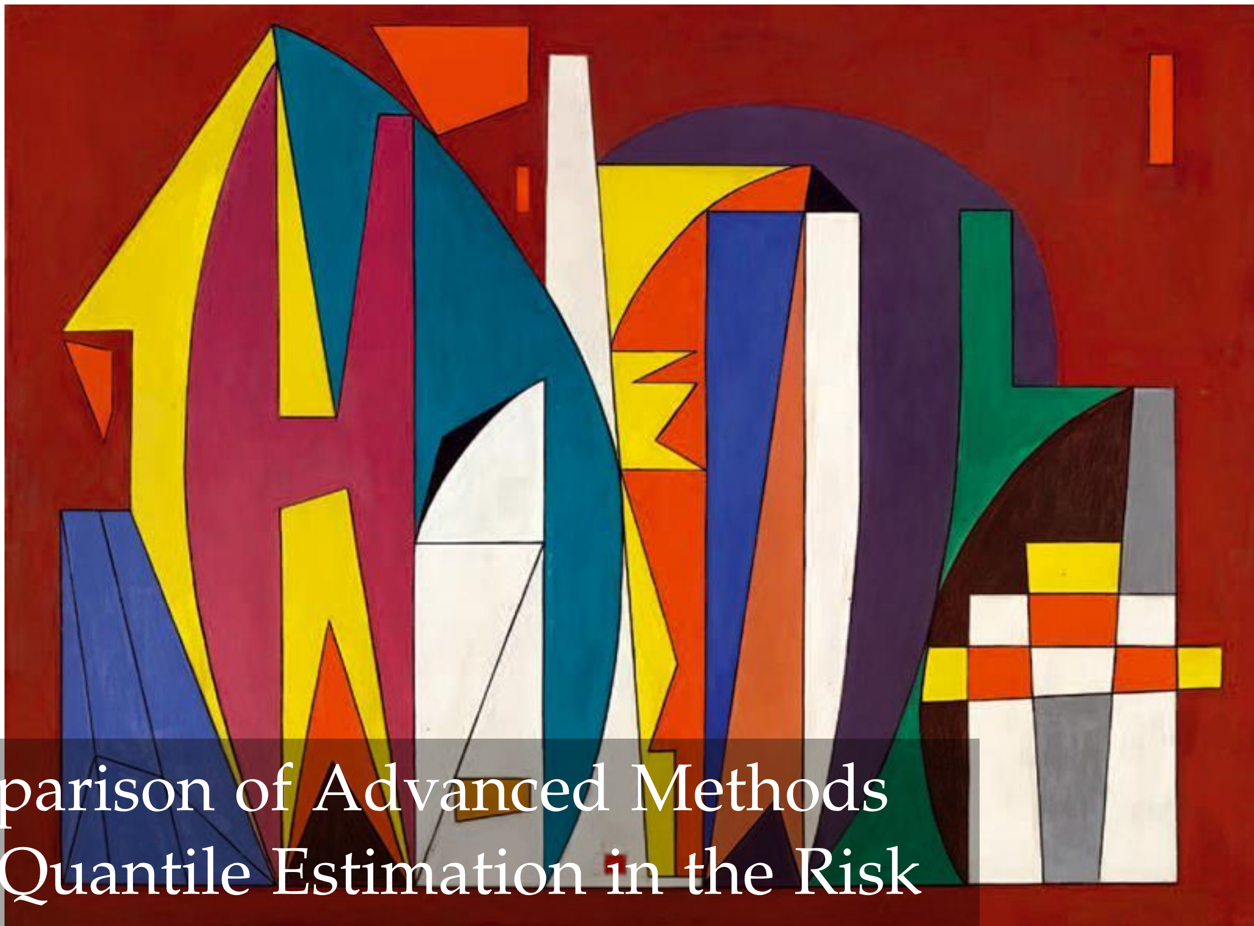

A Comparison of Advanced Methods for the Quantile Estimation in the Risk Management Field

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Executive Summary

In the financial sector the regulation prescribes what risk measures have to be implemented to guarantee a safe banking system. Due to the forthcoming new regulation about the market risk (FRTB), the banks have recently developed Montecarlo models to estimate the *Default Risk Charge* (DRC), namely a 1-year VaR at 99.9% confidence level related to the losses coming from the default events in the trading book. Banks have put a lot of effort in the modeling step of the process, i.e how to build the simulation algorithm for the process: what are the risk factors, how to define the default event, how to infer the correlations among the obligors and so on. Despite the extreme quantile estimation is a well-known problem in the statistical field, it has received less attention by the banks modelers. In our paper we review the context and the existing literature, hence we compare on real data the performance of some advanced quantile estimators for the DRC measure that could be used to challenge the classical empirical quantile. For small-medium size samples the results are encouraging.

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He holds a degree in mathematics, a master degree in mathematical finance at the Paris VII University and a PHD in statistics. He worked as an executive in both large and regional italian banks. He has also collaborated with some consultant companies in the broad area of risk management, asset management, pricing models, software systems and regulatory compliance. Along with his professional activities, he developed applied research in the above fields, with more than 20 papers published in scientific journals and dozens of speeches in international conferences. He currently gives seminars and lessons in some top ranked Italian universities.

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A Comparison of Advanced Methods for the Quantile Estimation in the Risk Management Field

Michele Bonollo Leonardo Mastrototaro

In recent years, the risk management field received an increasing attention, due to the growth in the banking regulation (the so-called Basel framework) that asked for more risk measures to the banks, to capture any potential source of risk (losses). Several quantitative disciplines were exploited to build more safe risk models, taking the techniques mainly from probability, stochastic processes, mathematical finance. Most of the attention of both the academic and financial community was devoted to build accurate models for the losses distributions, to define proper risk measures (e.g. the *Value at Risk* quantile vs. the *Expected Shortfall*), to decompose them to get a breakdown, to develop robust data quality steps for the market data. An exhaustive description of the risk management models is given in the following [9]. Some details about the risk measures breakdown are available in [5] and [6]. A recent deep analysis of the desirable properties of the risk measures is provided by [1]. Surpingsly, relative few attention was receveid by the last step of the risk management process, i.e. the measure estimation based on the empirical (or simulated) data. Given the model, given the data, it may happen that the risk estimation is not reliable, because of the usual uncertainty embedded in the historical or simulated data. This issue is very general, as it must be faced for any model (parametric or nonparametric) and for all risk measures (VaR, ES, ComponentVaR, etc.). In the forthcoming 'Basel IV' regulation the banks must estimate a very extreme quantile, namely a 99.9% quantile with 1 year horizon related to the default losses in the trading book, knwon as *DRC*, Default Risk Charge. Because of the lack of analytical models to calculate or to approximate satisfactory this quantile, all the banks adopt the Montecarlo simulation approach. Then we aim to compare some basic quantile estimators vs some more advanced tools, in order to check their performances in the bias-variance dimensions vs their computational complexity.

1. Financial Context. The DRC Risk Measure

The Default Risk Charge is a regulatory measure designed to capture default risk within the trading portfolio, as required by Basel standards, particularly within the framework of the Fundamental Review of the Trading Book (FRTB) outlined in the BCBS 457[2] document published by the Basel Committee on Banking Supervision. This model is designed to quantify the risk of loss resulting from the failure of a counterparty or issuer of financial instruments, including equity, bond, and derivative exposures. It is worth to note that banks could have *short* positions on the issuer debt, such as CDS, where the defaults imply a profit, not a loss. Then any DRC model requires a careful preliminary data management process where the granular positions related to each obligor are netted, aggregated, etc. The Default Risk Charge (DRC) regulatory set-up is described by Chapter 7 ('The Default Risk Charge') of the BCBS 457 document[2]. In the updated Basel Framework, the requirements related to the DRC are contained in section MAR 33, specifically in paragraphs MAR 33.18 - 33.38. These documents establish the criteria for calculating default risk, specifying that:

- It must be calculated over a one-year horizon.

- It must reflect a 99.9% threshold. The ECB EGIM [4] prescribes that the DRC measure must be provided with a confidence interval of 95%.
- It must include all exposures sensitive to default risk within the trading portfolio, excluding those specifically defined as "non-material risks".

The model aims to compute a portfolio loss distribution representative of overall credit risk. The DRC value is typically defined as the loss at the regulatory percentile (e.g., 99.9%) over a one-year horizon, with the loss defined as:

$$PnL = \sum_n EAD_n \cdot \mathbb{1}_{D_n} \cdot LGD_n \,, \tag{1}$$

where EAD_n is the exposure on the n-th instrument, D_n express the default status for the legal entity linked to that instrument and LGD_n is the loss given default.
The DRC simulation is implemented through a Monte Carlo approach, a probabilistic methodology that allows for a detailed modeling of uncertainty related to risk factors. Each iteration of the simulation includes:

- **Simulation of Credit Drivers**: Stochastic scenarios for credit risk factors are generated for each Legal Entity in the portfolio. This process simulates the defaults of individual Legal Entities.
- **Calculation of Recovery Rates**: For each scenario, the recovery rate associated with the simulated defaults of Legal Entities is estimated. This parameter is crucial for quantifying the recoverable loss amount.
- **Determination of PnL**: Using the simulated credit drivers and calculated recovery rates, the Profit and Loss (PnL) associated with each financial instrument in the portfolio is estimated, consolidating the results to obtain the overall loss distribution.

Further details and methodological applications can be found in Basel documentation, including BCBS 352 ("Standards: Minimum capital requirements for market risk") and the aforementioned BCBS 457[2], which provide detailed guidelines for the implementation and reporting of the DRC.

1.1 Credit Worthiness Index

The random variable X_i is the Credit Worthiness Index (CWI) and reflects the credit quality of legal entity i. It is simulated for each legal entity considered through a stochastic process defined as:

$$\Delta X_i = \frac{\sum_j \beta_{ij} \ \Delta Z_j}{\rho_i} + \frac{\sigma_i \ \Delta W_i}{\rho_i}, \tag{2}$$

where i refers to the legal entity, j refers to the credit drivers associated with the legal entity, Z_j is a multidimensional stochastic process of credit driver returns with mean 0 and covariance matrix C, W_i is a Gaussian variable $\mathcal{N}(0,1)$ independent from Z_j, β_{ij} are the credit drivers' weights, ρ is the total variance matrix of the Credit Worthiness process and ν and σ are the volatility matrix of the systemic component and idiosyncratic component, respectively.
Specifically, the processes $X_i = \sum \Delta X_i \Delta t$ simulate various risk drivers as a multivariate Gaussian distribution using the historical correlation matrix of the drivers, while the idiosyncratic component is simulated as a standard Gaussian distribution $\mathcal{N}(0,1)$ scaled by the volatility σ_i of the reference issuer. The CWI processes are used to determine the default of a given obligor. This occurs when X_i falls below a given threshold T_i, related to their rankings:

$$D_n = \{X_i \leq \Phi^{-1}(\mathrm{PD}_i)\}, \tag{3}$$

with Φ the cumulative distribution of a standard Gaussian and PD_i the probability of default of the i-th obligor.

1.2 Recovery Rates

Once defaults have been determined in each scenario, the recovery rate associated with each legal entity is computed. The model currently implemented assumes that the recovery rate process for obligor i, R_i, is identical to the CWI process:

$$R_i = \frac{X_i}{\rho_i}. \tag{4}$$

Finally, the final recovery rate value is obtained using the inverse of the Beta distribution:

$$\mathrm{r}_i = \beta_i^{-1}\left(F_i\right), \tag{5}$$

where F_i is a variable defined in $[0,1]$. The use of a β function is justified because the output is a variable in the range $[0,1]$. The parameters α and β of the Beta function are derived from input data and vary for each obligor. It is defined as:

$$F_i = \left(\frac{\Phi(\mathbf{R_i}/\sqrt{\Delta t})}{\Phi(¸\mathbf{i}/\sqrt{\Delta t})}\right), \tag{6}$$

where Φ is the cumulative Gaussian distribution, and ξ is the threshold determining the default of obligor i after time Δt. This is possible since the argument of the cumulative function corresponds to the CWI value, and the re-scaling ensures maximum recovery rate values when R_i equals the default threshold.

1.3 PnL Calculation

PnL is computed by aggregating the individual PnL for each instrument. The PnL for each instrument j is defined as:

$$PnL_j^S = \left(MtM_j - VaD_j^S\right) \times \mathbb{1}_{D_i}, \tag{7}$$

where MtM_j is the market value of the instrument, and VaD_j^S is the simulated value of instrument j and $\mathbb{1}_{D_i}$ takes into account the default of the legal entity i corresponding to instrument j. The actual calculation of VaD^S varies depending on the instrument and on the simulated recovery rate R_i^S. For example:

1. $VaD_j^S = MtM_j \times R_i^S$ if the instrument is a Bond;
2. $VaD_j^S = MtM_j \times (1 - R_i^S)$ if the instrument is a Credit Default Swap.

Once the PnL for each instrument j is computed, it is summed across all the instruments. This produces a PnL value for each scenario simulated and thus a vector with a length equal to the number of scenarios, whose 99.9th quantile represents the DRC value for the portfolio.

2. The Quantile Estimation

2.1 Review of Classical Results

The empirical quantile is used very often as a "plug and play" tool to estimate from the data the unknown true quantile, but at the end it is just one of the many approaches for the estimation purpose, exactly as the arithmetic median is an alternative to the classical arithmetic average to estimate an expected value. Let us introduce some simple notation. Let be X the random variable source of our data and suppose to have an i.i.d sample drawn from this distribution. We indicate with $X_{(n)}$ the n-th order statistics, i.e. the $n-th$ value after sorting by ascending order all the outcomes. Equipped with this notation, the distribution F_n of the n-th order statistics is given by:

$$F_{(n)}(x) = \Pr\{X_{(n)} \le x\} = \Pr\{\text{all } X_i \le x\} = F^n(x), \tag{8}$$

	Empirical	RF	TF	HD	EP	Empirical DRC
			10^4 Scenarios			
1st case	32.3%	16.7%	15.4%	14.8%	15.5%	311,050,198
2nd case	42.6%	36.4%	34.1%	29.1%	31.3%	97,262,380
3rd case	73.2%	36,2%	45,0%	41.2%	39.2%	189,501,850
			$2\text{x}10^5$ Scenarios			
1st case	6,66%	6,08%	5,96%	5,61%	5,98%	334,983,354
2nd case	23.4%	6.51%	6.37%	6.84%	6.34%	113,399,158
3rd case	35.5%	14.7%	14.3%	14.7%	14.2%	187,788,177
			$16\text{x}10^6$ Scenarios			
1st case	1.57%	1.44%	1.40%	1.37%	1.41%	329,265,406
2nd case	1.55%	1.38%	1.41%	1.68%	1.41%	112,998,640
3rd case	1.28%	1.16%	1.17%	1.30%	1.17%	186,601,892

TABLE 1: *Uncertainty for 3 different portfolio, with 10^4, 2×10^5 and 16×10^6 scenarios. In the table RF, TF, HD and EP stand respectively for Rectangular filter, triangular filter, Harrell-Davis estimator, Epanechnikov estimator. In the last column we have reported the empirical DRC value.*

while the empirical quantile $Q_n(\alpha)$ (supposing that the positive values represent the profits, the negative values the losses) writes as below:

$$Q_n(\alpha|X_1, ..., X_n) = X_{[(1-\alpha)\times n]}. \tag{9}$$

The integer part operator [] is needed to take an actual outcome from the sample.There some slightly different versions, according to less or more conservative (prudent) approaches. The most popular statistical tools (R,SAS, Matlab, Excel) also allow for different implementations of the quantile. A very relevant result for the order statistics is their asymptotic distribution. It can be shown that while the *min*() and the *max*() of the distribution never converge to the gaussian random variable, it happens for all the other order statistics. Namely we have the following result

$$E(Q_n(\alpha)) = Q(\alpha) - \frac{\alpha(1-\alpha)f'(Q(\alpha))}{2(n+2)f^3(Q(\alpha))} + O(1/n^2); \tag{10}$$

$$\text{Var}(Q_n(\alpha)) = \frac{\alpha(1-\alpha)}{(n+2)f^2(Q_n(\alpha))} + O(1/n^2). \tag{11}$$

A seminal reference in this filed is the textbook by[8] and in the work by [3]. If one analyzes the uncertainty of the empirical quantile as the parameters (n, α) change, one easily finds that te more extreme is the confidence level α and smaller is the sample size n, then less accurate (high variance) is the quantile estimator. Several attempts have been made in the inferential statistics field to define better estimators in the usual bias-variance trade-off.

2.2 Advanced Estimators from Order Statistics

We select some alternative estimators of the quantile as competitos of the basic empirical estimation. Considering the definition for an *L-estimator* as:

$$Q_n = \sum_i w_i X_i \ . \tag{12}$$

The *rectangular* filter is defined by $w_i = 1/n$ in an interval $[\alpha - \epsilon, \alpha + \epsilon]$.
The *triangular* filter is defined in the same interval with w_i maximum at α and symmetrically decreasing to 0 at $\alpha \pm \epsilon$.
Harrell-Davis estimator[7]:

$$w_i = I_{j/n}(a,b) - I_{(j-1)/n}(a,b)\,,$$

with

$$a = \alpha(n+1) \qquad b = (1-\alpha)(n+1) \qquad I_x(a,b) = \phi(\beta(a,b))\,,$$

where α is the confidence level, n total number of scenarios, ϕ cumulative distibution function e β is the beta Euler function.
Finally we have the *Epanechnikov* estimator:

$$w_i = K_{j/n}(\alpha,h) - K_{(j-1)/n}(\alpha,h),$$

where

$$K_x(\alpha,h) = \begin{cases} 0 & \text{if } x \le \alpha - h \\ \frac{1}{2} + \frac{3}{4}\frac{x-\alpha}{h} - \frac{1}{4}\left(\frac{x-\alpha}{h}\right)^3 & \text{if } \alpha - h < x < \alpha + h \\ 1 & \text{if } x \ge p + h \end{cases}$$

$h = 0.0005$.

3. Comparison of the Different Estimators

We extracted the PnL results from the DRC described in the previous section and applied the estimators presented in the "Introduction". The errors were assessed using the jackknife resampling method. Our findings indicate that the results obtained from the different estimators are consistent, with error estimates that remain comparable across 16 million scenarios. Indeed, for a sample portfolio composed of some thousands of positions (bonds, derivatives), belonging to about thousand obligors, we obtained the relative uncertainties expressed in Tab. 1. The uncertainty in the table represent the half width of a 95% confidence level of the quantile estimator. It is calculated by bootstrap approach for the more advanced tools, by the analytical results in Section "Review of Classical Results" for the empirical quantile. The same results are graphically expressed in Fig. 1 and 2.
The best result was obtained with the Harrell-Davis estimator[7], but the improvement is only 16% compared to the uncertainty of the empirical quantile. We analyzed these results by considering correlation values in the range $[0.9985, 0.9995]$. Specifically, we computed the correlation between the vector containing the elements at positions $[0.9985, 0.999]$ and the vector $[0.9985 + \epsilon, 0.999 + \epsilon]$, with $0 < \epsilon < 0.0005$. We obtained a mean correlation of 0.999, which indicates that, in this range, the points are highly correlated. Consequently, there is no significant difference whether we consider the position 0.999 directly or a nearby range. In other words, the correlation between any order statistics and the next one is so high that the smoothing technique embedded in the L-Estimators does not provide a relevant benefit in reducing the variance. Furthermore, since this is an extreme point, the range cannot be symmetrically extended beyond $\epsilon = 0.001$. To further support our analysis, we tested different distributions, extracting datasets of size 10^5, 10^6, and 10^7, and comparing the results. We considered both the normal distribution (not heavy-tailed) and the lognormal distribution (heavy-tailed), but we present here only the results for the lognormal distribution, as they are similar to those obtained with the normal distribution. We obtained the following absolute differences in percentage error:

- 10^5 points: $\sim$ 24%;
- 10^6 points: $\sim$ 10%;
- 10^7 points: $\sim$ 0.05%.

These values are computed as:

$$\frac{|\sigma(HD) - \sigma(\text{emp})|}{q}, \tag{13}$$

where q represents the empirical quantile value.
This result demonstrates that, beyond a certain dataset size, the different estimators yield the same uncertainty. Consequently, their adoption does not provide a significant advantage over the empirical quantile estimator.

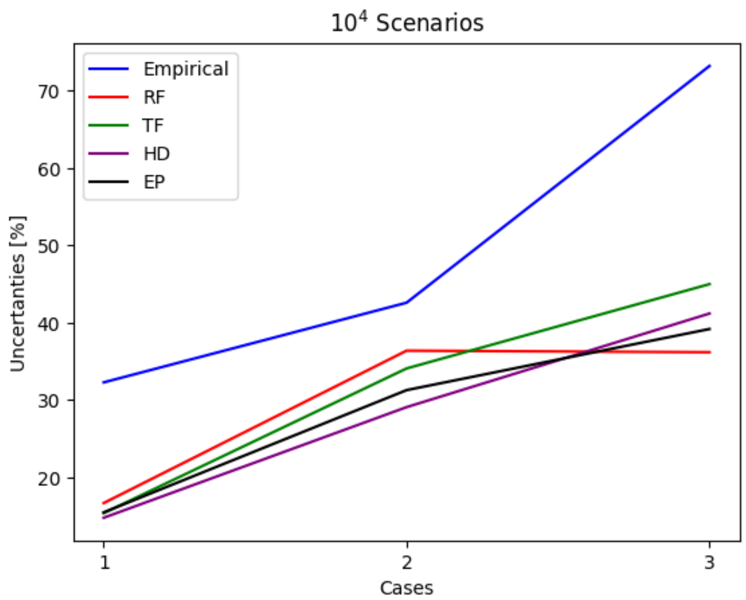

FIGURE 1: Uncertainty for 3 different portfolio, with 10^4 scenarios. In the legend RF, TF, HD and EP stand respectively for Rectangular filter, triangular filter, Harrell-Davis estimator, Epanechnikov estimator.

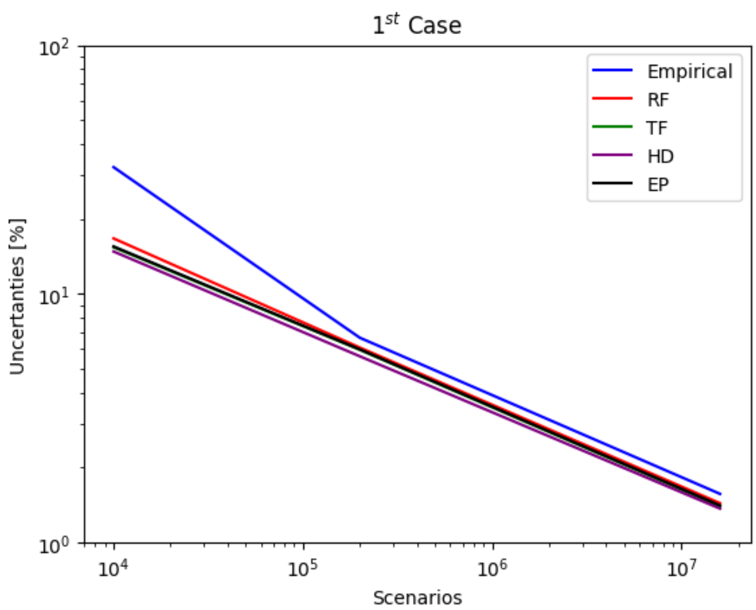

FIGURE 2: Uncertainty for different scenarios. In the legend RF, TF, HD and EP stand respectively for Rectangular filter, triangular filter, Harrell-Davis estimator, Epanechnikov estimator.

4. Conclusions

We exploit several techniques to face a very hard statistical problem, the estimation of an extreme quantile required by the banking regulations. The benchmark was the empirical quantile, some competitors come from classical theory, such as the filters, other were proposed by the statistical literature. We run some expercises on a real world large portfolio. We found that for a relative small number of simulations n, such as $O(n) = 10^4, 10^5$, the Harrel-Davis and the Epanechnikov methods show a relevant improvement in the variance reduction goal. when we can perform 10^6 or more simulations, the uncertainty of the more advanced tools becomes very close to the basic empirical quantile. To summarize, a bank should properly combine its hardware and software resources (and constraints) with its accuracy target, in order to achieve an adequate and sustainable risk measurement process.

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